

CONCENTRATING SOLUTIONS FOR A CLASS OF NONLINEAR FRACTIONAL SCHRÖDINGER EQUATIONS IN \mathbb{R}^N

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ABSTRACT. We deal with the existence of positive solutions for the following fractional Schrödinger equation

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = f(u) \text{ in } \mathbb{R}^N,$$

where $\varepsilon > 0$ is a parameter, $s \in (0, 1)$, $N > 2s$, $(-\Delta)^s$ is the fractional Laplacian operator, and $V : \mathbb{R}^N \rightarrow \mathbb{R}$ is a continuous positive function. Under the assumptions that the nonlinearity f is either asymptotically linear or superlinear at infinity, we prove the existence of solutions which concentrate around local minima of the potential $V(x)$.

1. INTRODUCTION

In this paper we investigate the existence and the concentration phenomenon of positive solutions for the following fractional equation

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = f(u) \text{ in } \mathbb{R}^N, \quad (1.1)$$

where $\varepsilon > 0$ is a parameter, $s \in (0, 1)$ and $N > 2s$. The external potential $V : \mathbb{R}^N \rightarrow \mathbb{R}$ is a locally Hölder continuous function, bounded below from zero, that is there exists $V_0 > 0$ such that

$$V(x) \geq V_0 > 0 \quad \text{for all } x \in \mathbb{R}^N. \quad (1.2)$$

Concerning the nonlinearity $f : \mathbb{R} \rightarrow \mathbb{R}$, we assume that it satisfies the following basic assumptions:

(f1) $f \in C^1(\mathbb{R}, \mathbb{R})$;

(f2) $\lim_{\xi \rightarrow 0} \frac{f(\xi)}{\xi} = 0$;

(f3) there exists $p \in (1, \frac{N+2s}{N-2s})$ such that $\lim_{\xi \rightarrow \infty} \frac{f(\xi)}{\xi^p} = 0$.

The nonlocal operator $(-\Delta)^s$ appearing in (1.1), is the so-called fractional Laplacian which can be defined for any $u : \mathbb{R}^N \rightarrow \mathbb{R}$ smooth enough, by setting

$$(-\Delta)^s u(x) = -\frac{C(N, s)}{2} \int_{\mathbb{R}^N} \frac{u(x+y) - u(x-y) - 2u(x)}{|y|^{N+2s}} dy \quad (x \in \mathbb{R}^N)$$

where $C(N, s)$ is a dimensional constant depending only on N and s ; see [22].

In the last decade, a great attention has been focused on the study of nonlinear elliptic problems involving fractional operators, due to their intriguing analytic structure and specially in view of several applications in many areas of the research such as crystal dislocation, finance, phase transitions, material sciences, chemical reactions, minimal surfaces. For more details and applications on this subject we refer the interested reader to [22, 38].

One of the main reason of the study of (1.1) is the search of standing wave solutions $\psi(t, x) = u(x)e^{-\frac{ict}{\hbar}}$ for the following time-dependent fractional Schrödinger equation

$$i\hbar \frac{\partial \Phi}{\partial t} = \frac{\hbar^2}{2m} (-\Delta)^s \Phi + W(x)\Phi - g(|\Phi|)\Phi \text{ for } (t, x) \in \mathbb{R} \times \mathbb{R}^N. \quad (1.3)$$

The equation (1.3) has been derived by Laskin in [34, 35], and plays a fundamental role in quantum mechanics in the study of particles on stochastic fields modeled by Lévy processes.

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When $s = 1$, the equation (1.1) becomes the classical Schrödinger equation

$$-\varepsilon^2 \Delta u + V(x)u = f(u) \text{ in } \mathbb{R}^N, \quad (1.4)$$

for which the existence and the multiplicity of solutions has been extensively studied in the last twenty years by many mathematicians; see [1, 3, 4, 11, 12, 29, 39, 41, 45].

Rabinowitz in [41] investigated the existence of positive solutions to (1.4) for $\varepsilon > 0$ small enough, under the assumption that f satisfies the well-known Ambrosetti-Rabinowitz condition [5], that is (f4) there exists $\mu > 2$ such that $0 < \mu F(t) \leq f(t)t$ for any $t > 0$, where $F(t) = \int_0^t f(\tau)d\tau$, and the potential $V(x)$ satisfies the following global condition

$$\liminf_{|x| \rightarrow \infty} V(x) > \inf_{x \in \mathbb{R}^N} V(x).$$

Wang [45] showed that these solutions concentrate at global minimum points of $V(x)$. By using a local mountain pass approach, Del Pino and Felmer in [21], proved the existence of a single spike solution to (1.4) which concentrate around a local minimum of V , by assuming that there exists a bounded open set Λ in \mathbb{R}^N such that

$$\inf_{x \in \Lambda} V(x) < \min_{x \in \partial \Lambda} V(x),$$

and considering nonlinearities f satisfying (f4) and the monotonicity assumption on $t \mapsto \frac{f(t)}{t}$. Subsequently, Jeanjean and Tanaka [33] introduced new variational methods to extend the results obtained in [21], to a wider class of nonlinearities.

Differently from the classic literature, in the non-local setting, there are even few results concerning the existence and the concentration phenomena of solutions for the fractional equation (1.1), maybe because many important techniques developed in local framework, cannot be adapted so easily to the fractional case.

In what follows, we recall some fundamental results related to the concentration phenomenon of solutions for the nonlinear fractional Schrödinger equation (1.1), obtained in recent years.

Chen and Zheng [17] studied via Liapunov-Schmidt reduction method, the concentration phenomenon for solutions of (1.1) with $f(t) = |t|^\alpha t$, and under suitable limitations on the dimension of the space N and the fractional powers s . Davila et al. [20] showed that if the potential V satisfies

$$V \in C^{1,\alpha}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \text{ and } \inf_{x \in \mathbb{R}^N} V(x) > 0,$$

then (1.1) has multi-peak solutions. Fall et al. [26] established necessary and sufficient conditions on the smooth potential V in order to produce concentration of solutions of (1.1) when the parameter ε converges to zero. In particular, when V is coercive and has a unique global minimum, then ground-states concentrate at this point. Shang and Zhang [43] proved the existence and the multiplicity of concentrating solutions for a class of fractional Schrödinger equations with vanishing potentials. Figueiredo and Siciliano [28] obtained a multiplicity result by means of the Lyusternik-Shnirelman and Morse theories for (1.1) involving a superlinear nonlinearity with subcritical growth. Alves and Miyagaki in [2] investigated the existence and the concentration of positive solutions to (1.1), via penalization approach. He and Zou [31] used variational methods and Ljusternik-Schnirelmann theory to study existence and multiplicity of solutions for a fractional Schrödinger equation involving the critical exponent. Finally, we would like also mention the papers [6, 7, 8, 9, 10, 16, 18, 19, 23, 24, 27, 30, 37, 40, 42, 44] in which the existence and the multiplicity of solutions for different nonlinear fractional Schrödinger equations has been investigated by using several variational approaches.

Motivated by the above papers, in this work we aim to study the existence of positive solutions to (1.1) concentrating around local minima of the potential $V(x)$, and under the assumptions that the nonlinearity f is asymptotically linear or superlinear at infinity. More precisely, our purpose is to extend the existence result in [33] for the nonlocal problem (1.1).

Now, we state our main result:

Theorem 1.1. *Let us assume that $f(\xi)$ satisfies (f1)-(f3) and either (f4), or the following condition (f5):*

- (i) *There exists $a \in (0, +\infty]$ such that $\lim_{\xi \rightarrow \infty} \frac{f(\xi)}{\xi} = a$;*
- (ii) *There exists $D \geq 1$ such that*

$$\hat{F}(s) \leq D\hat{F}(t) \quad 0 \leq s \leq t \quad (1.5)$$

where $\hat{F}(\xi) = \frac{1}{2}f(\xi)\xi - F(\xi)$.

Let $\Lambda \subset \mathbb{R}^N$ be a bounded open set such that

$$\inf_{\Lambda} V < \min_{\partial\Lambda} V \quad (1.6)$$

when $a < +\infty$ in (f5),

$$\inf_{\Lambda} V < a. \quad (1.7)$$

Then, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ the equation (1.1) admits a positive solution $u_\varepsilon(x)$. Moreover, if x_ε denotes the global maximum of u_ε , then we have

- (1) $V(x_\varepsilon) \rightarrow \inf_{x \in \Lambda} V(x)$;
- (2) there exists $C > 0$ such that

$$u_\varepsilon(x) \leq \frac{C\varepsilon^{N+2s}}{\varepsilon^{N+2s} + |x - x_\varepsilon|^{N+2s}} \quad \forall x \in \mathbb{R}^N.$$

A common approach to tackle fractional nonlocal problems, is make use of the extension method due to Caffarelli and Silvestre [15], which allows us to transform a given nonlocal equation into a degenerate elliptic problem in the half-space with a nonlinear Neumann boundary condition. In this work, we don't follow this approach, and we prefer to investigate the problem directly in $H^s(\mathbb{R}^N)$, in order to adapt in our framework some ideas developed in [33]. Clearly, due to the presence of the fractional Laplacian $(-\Delta)^s$, which is a nonlocal operator, more accurate estimates are needed.

We would like to note that Theorem 1.1 extends and improves the result in [2], because we don't require any monotonicity assumption on $f(t)/t$, and we are able to deal with a more general class of nonlinearities, including the asymptotically linear case. Moreover, our result is in clear accordance with that for the classical local counterpart, that is Theorem 1.1 in [33].

We also point out that in contrast with the case $s = 1$, the decay at infinity of solutions of (1.1) is of power-type and not exponential; see [27].

Now, we give the main ideas for the proof of Theorem 1.1. After rescaled the equation (1.1) with the change of variable $v(x) = u(\varepsilon x)$, we introduce a modified functional J_ε and we prove that it satisfies a mountain pass geometry. Then, we investigate the boundedness of Cerami sequences for J_ε , and we give two types of boundedness results: one when ε is fixed, the other one to deduce uniform boundedness when $\varepsilon \rightarrow 0$. Through a careful analysis of the behavior as $\varepsilon \rightarrow 0$ of bounded Cerami sequences (v_ε) , we prove that there exists a subsequence (v_{ε_j}) which converges, in a suitable sense, to a sum of translated critical points of certain autonomous functionals. This concentration-compactness type result, will be useful to show that an appropriate translated sequence $v_{\varepsilon_j}(\cdot + y_{\varepsilon_j})$ converges to a least energy solution ω^1 . Then, we exploit some results obtained in [27], to deduce L^∞ -estimates (uniformly in $j \in \mathbb{N}$) and some informations about the behavior at infinity of the translated sequence, which permit to obtain a positive solution of the rescaled equation.

The body of the paper is the following: in Section 2 we collect some preliminary results concerning the fractional Sobolev spaces and we introduce the variational setting. In Section 3 we study the modified functionals J_ε . In Section 4 we present some fundamental properties related to autonomous functionals. In Section 5 we give a concentration-compactness type result. In the last section we provide the proof of Theorem 1.1.

2. PRELIMINARIES AND FUNCTIONAL SETTING

2.1. Fractional Sobolev spaces and some useful Lemmas. In this section, we briefly recall some properties of the fractional Sobolev spaces, and we introduce some notations which we will use along the paper.

For any $s \in (0, 1)$, we denote by $\mathcal{D}^{s,2}(\mathbb{R}^N)$ the completion of the set $C_0^\infty(\mathbb{R}^N)$ consisting of the infinitely differentiable functions $u : \mathbb{R}^N \rightarrow \mathbb{R}$ with compact support, with respect to the following norm

$$[u]^2 = \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy = \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^N)}^2.$$

Equivalently,

$$\mathcal{D}^{s,2}(\mathbb{R}^N) = \left\{ u \in L^{2^*}(\mathbb{R}^N) : [u] < \infty \right\}.$$

Now, let us define the fractional Sobolev space

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \frac{|u(x) - u(y)|}{|x - y|^{\frac{N+2s}{2}}} \in L^2(\mathbb{R}^{2N}) \right\}$$

endowed with the natural norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \sqrt{[u]^2 + \|u\|_{L^2(\mathbb{R}^N)}^2}.$$

For the convenience of the reader we recall the following fundamental embeddings:

Theorem 2.1. [22] *Let $s \in (0, 1)$ and $N > 2s$. Then there exists a sharp constant $S_* = S(N, s) > 0$ such that for any $u \in H^s(\mathbb{R}^N)$*

$$\|u\|_{L^{2^*}(\mathbb{R}^N)}^2 \leq S_* [u]^2. \quad (2.1)$$

Moreover $H^s(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ for any $q \in [2, 2_s^]$ and compactly in $L_{loc}^q(\mathbb{R}^N)$ for any $q \in [2, 2_s^*)$.*

Now, we prove the following technical result which will be useful in the sequel.

Lemma 2.1. (i) *Let $(w_j) \subset H^s(\mathbb{R}^N)$ be a bounded sequence in $H^s(\mathbb{R}^N)$, and let $\phi \in C_0^\infty(\mathbb{R}^N)$ be a function such that $0 \leq \phi \leq 1$, $\phi = 1$ in B_1 , $\phi = 0$ in $\mathbb{R}^N - B_2$. Set $\phi^\varepsilon(x) = \phi(\frac{x}{\varepsilon})$. Then, we get*

$$\lim_{\varepsilon \rightarrow 0} \lim_{j \rightarrow \infty} \iint_{\mathbb{R}^{2N}} w_j^2(x) \frac{(\phi^\varepsilon(x) - \phi^\varepsilon(y))^2}{|x - y|^{N+2s}} dx dy = 0.$$

(ii) *Let $(w_j) \subset H^s(\mathbb{R}^N)$ be a bounded sequence in $H^s(\mathbb{R}^N)$, and let $\eta \in C^\infty(\mathbb{R}^N)$ be a function such that $0 \leq \eta \leq 1$, $\eta = 0$ in B_1 , $\eta = 1$ in $\mathbb{R}^N - B_2$. Set $\eta_R(x) = \eta(\frac{x}{R})$. Then, we get*

$$\lim_{R \rightarrow \infty} \lim_{j \rightarrow \infty} \iint_{\mathbb{R}^{2N}} w_j^2(x) \frac{(\eta_R(x) - \eta_R(y))^2}{|x - y|^{N+2s}} dx dy = 0.$$

Proof. We begin proving (i). Let us note that \mathbb{R}^{2N} can be written as

$$\mathbb{R}^{2N} = ((\mathbb{R}^N - B_{2\varepsilon}) \times (\mathbb{R}^N - B_{2\varepsilon})) \cup (B_{2\varepsilon} \times \mathbb{R}^N) \cup ((\mathbb{R}^N - B_{2\varepsilon}) \times B_{2\varepsilon}) =: X_\varepsilon^1 \cup X_\varepsilon^2 \cup X_\varepsilon^3.$$

Then

$$\begin{aligned} & \iint_{\mathbb{R}^{2N}} w_j^2(x) \frac{(\phi^\varepsilon(x) - \phi^\varepsilon(y))^2}{|x - y|^{N+2s}} dx dy \\ &= \iint_{X_\varepsilon^1} w_j^2(x) \frac{|\phi^\varepsilon(x) - \phi^\varepsilon(y)|^2}{|x - y|^{N+2s}} dx dy + \iint_{X_\varepsilon^2} w_j^2(x) \frac{|\phi^\varepsilon(x) - \phi^\varepsilon(y)|^2}{|x - y|^{N+2s}} dx dy \\ &+ \iint_{X_\varepsilon^3} w_j^2(x) \frac{|\phi^\varepsilon(x) - \phi^\varepsilon(y)|^2}{|x - y|^{N+2s}} dx dy. \end{aligned} \quad (2.2)$$

In what follows, we estimate each integral in (2.2). Since $\phi = 0$ in $\mathbb{R}^N \setminus B_2$, we have

$$\iint_{X_\varepsilon^1} w_j^2(x) \frac{|\phi_\varepsilon(x) - \phi_\varepsilon(y)|^2}{|x - y|^{N+2s}} dx dy = 0. \quad (2.3)$$

Now, by using $0 \leq \phi \leq 1$, we can see that

$$\begin{aligned} & \iint_{X_\varepsilon^2} w_j^2(x) \frac{|\phi^\varepsilon(x) - \phi^\varepsilon(y)|^2}{|x - y|^{N+2s}} dx dy \\ &= \int_{B_{2\varepsilon}} dx \int_{\{y \in \mathbb{R}^N : |x-y| \leq \varepsilon\}} w_j^2(x) \frac{|\phi^\varepsilon(x) - \phi^\varepsilon(y)|^2}{|x - y|^{N+2s}} dy + \int_{B_{2\varepsilon}} dx \int_{\{y \in \mathbb{R}^N : |x-y| > \varepsilon\}} w_j^2(x) \frac{|\phi^\varepsilon(x) - \phi^\varepsilon(y)|^2}{|x - y|^{N+2s}} dy \\ &\leq \varepsilon^{-2} |\nabla \phi|_{L^\infty(\mathbb{R}^N)}^2 \int_{B_{2\varepsilon}} dx \int_{\{y \in \mathbb{R}^N : |x-y| \leq \varepsilon\}} \frac{w_j^2(x)}{|x - y|^{N+2s-2}} dy + 4 \int_{B_{2\varepsilon}} dx \int_{\{y \in \mathbb{R}^N : |x-y| > \varepsilon\}} \frac{w_j^2(x)}{|x - y|^{N+2s}} dy \\ &\leq c_1 \varepsilon^{-2s} \int_{B_{2\varepsilon}} w_j^2(x) dx + c_2 \varepsilon^{-2s} \int_{B_{2\varepsilon}} w_j^2(x) dx = c_3 \varepsilon^{-2s} \int_{B_{2\varepsilon}} w_j^2(x) dx \end{aligned} \quad (2.4)$$

for some $c_1, c_2, c_3 > 0$ independent of ε and j .

On the other hand

$$\begin{aligned} & \iint_{X_\varepsilon^3} w_j^2(x) \frac{|\phi^\varepsilon(x) - \phi^\varepsilon(y)|^2}{|x - y|^{N+2s}} dx dy \\ &= \int_{\mathbb{R}^N \setminus B_{2\varepsilon}} dx \int_{\{y \in B_{2\varepsilon} : |x-y| \leq \varepsilon\}} w_j^2(x) \frac{|\phi^\varepsilon(x) - \phi^\varepsilon(y)|^2}{|x - y|^{N+2s}} dy \\ &+ \int_{\mathbb{R}^N \setminus B_{2\varepsilon}} dx \int_{\{y \in B_{2\varepsilon} : |x-y| > \varepsilon\}} w_j^2(x) \frac{|\phi^\varepsilon(x) - \phi^\varepsilon(y)|^2}{|x - y|^{N+2s}} dy =: A_{\varepsilon,j} + B_{\varepsilon,j}. \end{aligned} \quad (2.5)$$

Now, we note that

$$\begin{aligned} A_{\varepsilon,j} &\leq \varepsilon^{-2} |\nabla \phi|_{L^\infty(\mathbb{R}^N)}^2 \int_{B_{3\varepsilon}} dx \int_{\{y \in \mathbb{R}^N : |x-y| \leq \varepsilon\}} \frac{w_j^2(x)}{|x - y|^{N+2s-2}} dy \\ &\leq \varepsilon^{-2} |\nabla \phi|_{L^\infty(\mathbb{R}^N)}^2 \alpha_{N-1} \int_{B_{3\varepsilon}} w_j^2(x) dx \int_{\{z \in \mathbb{R}^N : |z| \leq \varepsilon\}} \frac{1}{|z|^{N+2s-2}} dz \\ &= c_4 \varepsilon^{-2s} \int_{B_{3\varepsilon}} w_j^2(x) dx \end{aligned} \quad (2.6)$$

for some $c_4 > 0$ independent of ε and j . Here α_{N-1} is the Lebesgue measure of the unit sphere in \mathbb{R}^N . Let us observe, that there exists $K > 4$ such that

$$(\mathbb{R}^N \setminus B_{2\varepsilon}) \times B_{2\varepsilon} \subset (B_{K\varepsilon} \times B_{2\varepsilon}) \cup ((\mathbb{R}^N \setminus B_{K\varepsilon}) \times B_{2\varepsilon}).$$

Then, we have the following estimates

$$\begin{aligned} & \int_{B_{K\varepsilon}} dx \int_{\{y \in B_{2\varepsilon} : |x-y| > \varepsilon\}} w_j^2(x) \frac{|\phi^\varepsilon(x) - \phi^\varepsilon(y)|^2}{|x - y|^{N+2s}} dy \\ &\leq 4 \int_{B_{K\varepsilon}} dx \int_{\{y \in B_{2\varepsilon} : |x-y| > \varepsilon\}} w_j^2(x) \frac{1}{|x - y|^{N+2s}} dy \\ &\leq 4\alpha_{N-1} \int_{B_{K\varepsilon}} dx \int_{\{z \in \mathbb{R}^N : |z| > \varepsilon\}} w_j^2(x) \frac{1}{|z|^{N+2s}} dz \\ &= c_5 \varepsilon^{-2s} \int_{B_{K\varepsilon}} w_j^2(x) dx. \end{aligned} \quad (2.7)$$

for some c_5 independent of ε and j , and

$$\begin{aligned}
& \int_{\mathbb{R}^N \setminus B_{K\varepsilon}} dx \int_{\{y \in B_{2\varepsilon} : |x-y| > \varepsilon\}} w_j^2(x) \frac{|\phi^\varepsilon(x) - \phi^\varepsilon(y)|^2}{|x-y|^{N+2s}} dy \\
& \leq 2^{N+2s} 4 \int_{\mathbb{R}^N \setminus B_{K\varepsilon}} dx \int_{\{y \in B_{2\varepsilon} : |x-y| > \varepsilon\}} \frac{w_j^2(x)}{|x|^{N+2s}} dy \\
& \leq 2^{N+2s} 4 (2\varepsilon)^N \int_{\mathbb{R}^N \setminus B_{K\varepsilon}} \frac{w_j^2(x)}{|x|^{N+2s}} dx \\
& \leq 2^{N+2s} 4 (2\varepsilon)^N \left(\int_{\mathbb{R}^N \setminus B_{K\varepsilon}} w_j^{2^*}(x) dx \right)^{\frac{2}{2^*}} \left(\int_{\mathbb{R}^N \setminus B_{K\varepsilon}} |x|^{-(N+2s)\frac{2^*}{2^*-2}} dx \right)^{\frac{2^*-2}{2^*}} \\
& \leq c_6 K^{-N} \left(\int_{\mathbb{R}^N \setminus B_{K\varepsilon}} w_j^{2^*}(x) dx \right)^{\frac{2}{2^*}}
\end{aligned} \tag{2.8}$$

for some $c_6 > 0$ independent of ε and j . Taking into account (2.7) and (2.8), and the fact that (w_j) is bounded in $L^{2^*}_s(\mathbb{R}^N)$, we can find $c_7 > 0$ independent of ε and j such that

$$B_{\varepsilon,j} \leq c_5 \varepsilon^{-2s} \int_{B_{K\varepsilon}} w_j^2(x) dx + c_8 K^{-N}. \tag{2.9}$$

Therefore, putting together (2.2)-(2.6) and (2.9), we can see that

$$\iint_{\mathbb{R}^{2N}} w_j^2(x) \frac{|\phi^\varepsilon(x) - \phi^\varepsilon(y)|^2}{|x-y|^{N+2s}} dx dy \leq c_7 \varepsilon^{-2s} \int_{B_{K\varepsilon}} w_j^2(x) dx + c_8 K^{-N} \tag{2.10}$$

for some $c_7, c_8 > 0$ independent of ε and j . Since (w_j) is bounded in $H^s(\mathbb{R}^N)$, we may assume (see Theorem 2.1) that $w_j \rightharpoonup w$ in $H^s(\mathbb{R}^N)$ and strongly in $L^2_{loc}(\mathbb{R}^N)$. Hence, we can deduce that

$$\lim_{j \rightarrow \infty} c_7 \varepsilon^{-2s} \int_{B_{K\varepsilon}} w_j^2(x) dx + c_8 K^{-N} = c_7 \varepsilon^{-2s} \int_{B_{K\varepsilon}} w^2(x) dx + c_8 K^{-N}.$$

Moreover, by using Hölder inequality, we have

$$\begin{aligned}
c_7 \varepsilon^{-2s} \int_{B_{K\varepsilon}} w^2(x) dx + c_8 K^{-N} & \leq c_7 \varepsilon^{-2s} \left(\int_{B_{K\varepsilon}} w^2(x) dx \right)^{\frac{2}{2^*}} |B_{K\varepsilon}(0)|^{1-\frac{2}{2^*}} + c_8 K^{-N} \\
& \leq c_9 K^{2s} \left(\int_{B_{K\varepsilon}} w^2(x) dx \right)^{\frac{2}{2^*}} + c_8 K^{-N} \rightarrow c_8 K^{-N} \text{ as } \varepsilon \rightarrow 0.
\end{aligned}$$

Then, we can infer that

$$\begin{aligned}
& \limsup_{\varepsilon \rightarrow 0} \limsup_{j \rightarrow \infty} \iint_{\mathbb{R}^{2N}} w_j^2(x) \frac{|\phi^\varepsilon(x) - \phi^\varepsilon(y)|^2}{|x-y|^{N+2s}} dx dy \\
& = \lim_{K \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \limsup_{j \rightarrow \infty} \iint_{\mathbb{R}^{2N}} w_j^2(x) \frac{|\phi^\varepsilon(x) - \phi^\varepsilon(y)|^2}{|x-y|^{N+2s}} dx dy = 0.
\end{aligned}$$

The proof of (ii) can be obtained in similar way, observing that

$$\frac{|\eta_R(x) - \eta_R(y)|^2}{|x-y|^{N+2s}} = \frac{|(1 - \eta_R(x)) - (1 - \eta_R(y))|^2}{|x-y|^{N+2s}},$$

and that $1 - \eta$ satisfies the same assumptions of ϕ . □

Now, let us define the space of radial functions in $H^s(\mathbb{R}^N)$

$$H_r^s(\mathbb{R}^N) = \{u \in H^s(\mathbb{R}^N) : u(x) = u(|x|)\}.$$

Related to this space, the following compactness result due to Lions [36] holds:

Theorem 2.2. [36] *Let $s \in (0, 1)$ and $N > 2s$. Then $H_r^s(\mathbb{R}^N)$ is compactly in $L^q(\mathbb{R}^N)$ for any $q \in (2, 2_s^*)$.*

Finally, we recall the following useful lemmas:

Lemma 2.2. [10] *Let $N > 2s$ and $r \in [2, 2_s^*)$. If (u_j) is a bounded sequence in $H^s(\mathbb{R}^N)$ and if*

$$\lim_{j \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n|^r dx = 0$$

where $R > 0$, then $u_j \rightarrow 0$ in $L^t(\mathbb{R}^N)$ for all $t \in (2, 2_s^)$.*

Lemma 2.3. [16] *Let $(X, \|\cdot\|)$ be a Banach space such that X is embedded respectively continuously and compactly into $L^q(\mathbb{R}^N)$ for $q \in [q_1, q_2]$ and $q \in (q_1, q_2)$, where $q_1, q_2 \in (0, \infty)$. Assume that $(u_j) \subset X$, $u : \mathbb{R}^N \rightarrow \mathbb{R}$ is a measurable function and $P \in C(\mathbb{R}, \mathbb{R})$ is such that*

- (i) $\lim_{|t| \rightarrow 0} \frac{P(t)}{|t|^{q_1}} = 0$,
- (ii) $\lim_{|t| \rightarrow \infty} \frac{P(t)}{|t|^{q_2}} = 0$,
- (iii) $\sup_{j \in \mathbb{N}} \|u_j\|_X < \infty$,
- (iv) $\lim_{j \rightarrow \infty} P(u_j(x)) = u(x)$ for a.e. $x \in \mathbb{R}^N$.

Then, up to a subsequence, we have

$$\lim_{j \rightarrow \infty} \|P(u_j) - u\|_{L^1(\mathbb{R}^N)} = 0.$$

2.2. Modification of the nonlinearity. Since we look for positive solutions of (1.1), we can suppose that $f(\xi) = 0$ for any $\xi \leq 0$.

We recall the following useful properties of the function f :

Lemma 2.4. [33] *Assume that (f1)-(f3) hold. Then we have*

(i) *For all $\delta > 0$ there exists $C_\delta > 0$ such that*

$$|f(t)| \leq \delta |t| + C_\delta |t|^p \quad \forall t \in \mathbb{R}; \quad (2.11)$$

(ii) *If (f4) holds, then $f(t) \geq 0$ for all $t \geq 0$;*

(iii) *If (f5) holds, then $f(t) \geq 0$, $(\hat{F})(t) \geq 0$, $\frac{d}{dt}(\frac{F(t)}{t^2}) \geq 0$ for all $t \geq 0$;*

(iv) *If $t \mapsto \frac{f(t)}{t}$ is nondecreasing for $t \in (0, \infty)$, then (f5) is satisfied with $D = 1$.*

Let us suppose that $f(t)$ satisfies (f1)-(f3) and $V_0 < a = \lim_{\xi \rightarrow \infty} \frac{f(t)}{t} \in (0, +\infty]$.

We take $\nu \in (0, \frac{V_0}{2})$ and we define

$$\underline{f}(t) := \begin{cases} \min\{f(t), \nu t\} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0. \end{cases}$$

By using (f2) we can see that there exists $r_\nu > 0$ such that

$$\underline{f}(t) = f(t) \quad \forall |t| \leq r_\nu$$

In fact, taking $\varepsilon = \nu$ there exists r_ν such that

$$|t| \leq r_\nu \Rightarrow |f(t)| < \nu |t|$$

and by the definition of $\underline{f}(t)$, we can deduce that $\underline{f}(t) = f(t)$ for all $|t| \leq r_\nu$.
Now, we choose ν as follows

(1) If (f4) holds, then we take $\nu > 0$ such that

$$\frac{\nu}{2V_0} < \frac{1}{2} - \frac{1}{\mu}; \quad (2.12)$$

(2) If (f5) holds, then we take $\nu \in (0, \frac{V_0}{2})$ such that ν is a regular value of $t \mapsto \frac{f(t)}{t}$. Since $\lim_{t \rightarrow 0} \frac{f(t)}{t} = 0$ and $\lim_{t \rightarrow \infty} \frac{f(t)}{t} = a > V_0 > \nu$, if ν is a regular value of $\frac{f(t)}{t}$ we deduce that $k_\nu = \text{card}\{t \in (0, +\infty) : f(t) = \nu t\} < +\infty$.

Let $\Lambda \subset \mathbb{R}^N$ be a bounded open set such that $\partial\Lambda \in C^\infty$, and we assume that Λ verifies (1.6). Take $\Lambda' \subset \Lambda$ be an open set such that $\partial\Lambda'$ is of class C^∞ , and we define $\chi \in C^\infty(\mathbb{R}^N, \mathbb{R})$ such that

$$\begin{aligned} \inf_{\Lambda - \Lambda'} V &> \inf_{\Lambda} V, \\ \min_{\partial\Lambda'} V &> \inf_{\Lambda'} V = \inf_{\Lambda} V, \\ \chi(x) &= 1 \quad \forall x \in \Lambda', \\ \chi(x) &\in (0, 1) \quad \forall x \in \Lambda - \Lambda', \\ \chi(x) &= 0 \quad \forall x \in \mathbb{R}^N - \Lambda. \end{aligned}$$

We suppose that $0 \in \Lambda'$ and $V(0) = \inf_{x \in \Lambda} V(x)$.

Let us introduce

$$g(x, t) = \chi(x)f(t) + (1 - \chi(x))\underline{f}(x) \quad \forall (x, t) \in \mathbb{R}^N \times \mathbb{R} \quad (2.13)$$

and we set

$$\begin{aligned} \underline{F}(t) &= \int_0^t \underline{f}(\tau) d\tau \\ G(x, t) &= \int_0^t g(x, \tau) d\tau = \chi(x)F(t) + (1 - \chi(x))\underline{F}(t). \end{aligned}$$

We recall some useful properties concerning $\underline{f}(t)$ and $g(x, t)$.

Lemma 2.5. [33]

- (i) $\underline{f}(t) = 0$, $\underline{F}(t) = 0$ for all $t \leq 0$;
- (ii) $\underline{f}(t) \leq f(t)$ for all $t \geq 0$;
- (iii) $\underline{f}(t) \leq \nu t$, $\underline{F}(t) \leq F(t)$ for all $t \geq 0$;
- (iv) If $f(t)$ satisfies (f4) or (f5) then $\underline{f}(t) \geq 0$ for all $t \in \mathbb{R}$;
- (v) If $f(t)$ verifies (f5), then $\underline{f}(t)$ satisfies (f5). Moreover, $\hat{\underline{F}}(t) \geq 0$ for all $t \geq 0$.

Corollary 2.1. [33]

- (i) $g(x, t) \leq f(t)$, $G(x, t) \leq F(t)$ for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}$;
- (ii) $g(x, t) = f(t)$ if $|t| < r_\nu$;
- (iii) For any $\delta > 0$ there exists $C_\delta > 0$ such that

$$|g(x, t)| \leq \delta|t| + C_\delta|t|^p \quad \text{for all } (x, t) \in \mathbb{R}^N \times \mathbb{R};$$

- (iv) if $f(t)$ satisfies (f5)- (ii) then $g(x, t)$ verifies

$$\hat{G}(x, \xi) \leq D^{k_\nu} \hat{G}(x, t) \quad \text{for all } 0 \leq \xi \leq t$$

where $\hat{G}(x, t) = \frac{1}{2}g(x, t)t - G(x, t)$, $D \geq 1$ is given by (f5)- (ii) and $k_\nu = \text{card}\{t \in (0, +\infty) : f(t) = \nu t\}$.

In what follows, we investigate the existence of positive solutions u_ε of the following problem

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = g(x, u) \quad \text{in } \mathbb{R}^N \quad (2.14)$$

via mountain-pass argument. Moreover, we are able to show that, for $\varepsilon > 0$ small enough, a solution u_ε of (2.14) satisfies $|u_\varepsilon(x)| < r_\nu$ for $x \in \mathbb{R}^N - \Lambda'$, that is, by using the definition of g , u_ε is a solution of (1.1).

2.3. Mountain pass argument. Let us introduce the rescaled function $v(x) = u(\varepsilon x)$, and we note that (2.14) becomes

$$(-\Delta)^s v + V(\varepsilon x)v = g(\varepsilon x, v) \quad \text{in } \mathbb{R}^N \quad (2.15)$$

From now on, we will be interested in the existence of solutions to (2.15). The functional energy associated to (2.15), is given by

$$J_\varepsilon(v) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v|^2 + V(\varepsilon x)v^2 dx - \int_{\mathbb{R}^N} G(\varepsilon x, v) dx \quad \forall v \in H_\varepsilon$$

where the fractional space

$$H_\varepsilon^s = \left\{ v \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(\varepsilon x)v^2 dx < \infty \right\}$$

is endowed with the norm

$$\|v\|_{H_\varepsilon^s}^2 = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v|^2 + V(\varepsilon x)v^2 dx.$$

Since $V_0 > 0$, we can endow $H^s(\mathbb{R}^N)$ with the following equivalent norm

$$\|v\|_{H^s(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v|^2 + V_0 v^2 dx.$$

Clearly,

$$\|v\|_{H^s} \leq \|v\|_{H_\varepsilon^s} \quad (2.16)$$

so we get $H_\varepsilon^s \subset H^s(\mathbb{R}^N)$ and H_ε^s is continuously embedded into $L^r(\mathbb{R}^N)$ for any $2 \leq r \leq \frac{2N}{N-2s}$, that is there exists $C'_r > 0$ such that

$$\|v\|_{L^r(\mathbb{R}^N)} \leq C'_r \|v\|_{H_\varepsilon^s}. \quad (2.17)$$

We start proving that J_ε possesses a mountain pass geometry.

Lemma 2.6. $J_\varepsilon \in C^1(H_\varepsilon^s, \mathbb{R})$ and verifies the following properties:

- (i) $J_\varepsilon(0) = 0$;
- (ii) there exist $\rho_0 > 0$ e $\delta_0 > 0$ independent of $\varepsilon \in (0, 1]$ such that

$$\begin{aligned} J_\varepsilon(v) &\geq \delta_0 \text{ for all } \|v\|_{H^1} = \rho_0 \\ J_\varepsilon(v) &> 0 \text{ for all } 0 < \|v\|_{H^1} \leq \rho_0; \end{aligned}$$

- (iii) there exists $v_0 \in C_0^\infty(\mathbb{R}^N)$ and $\varepsilon_0 > 0$ such that $J_\varepsilon(v_0) < 0$ for all $\varepsilon \in (0, \varepsilon_0]$.

Proof. Obviously, $J_\varepsilon \in C^1(H_\varepsilon^s, \mathbb{R})$ and $J_\varepsilon(0) = 0$ (because of $G(\varepsilon x, 0) = 0$). By using $\underline{F} \leq F$ and taking $\delta = \frac{V_0}{2}$ in Lemma 2.4, we get

$$\begin{aligned} J_\varepsilon(v) &= \frac{1}{2} \|v\|_{H_\varepsilon^s}^2 - \int_{\mathbb{R}^N} \chi(\varepsilon x) F(v) + (1 - \chi(\varepsilon x)) \underline{F}(v) dx \\ &\geq \frac{1}{2} \|v\|_{H_\varepsilon^s}^2 - \int_{\mathbb{R}^N} F(v) dx \\ &\geq \frac{1}{2} \|v\|_{H_\varepsilon^s}^2 - \int_{\mathbb{R}^N} \delta \frac{|v|^2}{2} + C_\delta \frac{|v|^{p+1}}{s+1} dx \\ &\geq \frac{1}{2} \|v\|_{H^s(\mathbb{R}^N)}^2 - \frac{V_0}{4} \|v\|_{L^2(\mathbb{R}^N)}^2 - C_{\frac{V_0}{2}} \|v\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} \\ &\geq \frac{\|v\|_{H^s(\mathbb{R}^N)}^2}{4} - \tilde{C}_{p+1} C_{\frac{V_0}{2}} \|v\|_{H^s(\mathbb{R}^N)}^{p+1} \end{aligned}$$

where in the last inequality we used (2.17) with $r = p + 1$. Then there exist $\rho_0, \delta_0 > 0$ such that (ii) is satisfied.

In order to verify that (iii) holds, we first note that in Section 5 (see Proposition 5.2), we will prove that

$$v \mapsto \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v|^2 + V(0)v^2 dx - \int_{\mathbb{R}^N} F(v) dx$$

has a mountain pass geometry, so we can take $v_0 \in C_0^\infty(\mathbb{R}^N)$ such that

$$\frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v_0|^2 + V(0)v_0^2 dx - \int_{\mathbb{R}^N} F(v_0) dx < 0.$$

Since $0 \in \Lambda'$ we can observe that

$$J_\varepsilon(v_0) \rightarrow \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v_0|^2 + V(0)v_0^2 dx - \int_{\mathbb{R}^N} F(v_0) dx < 0 \text{ as } \varepsilon \rightarrow 0,$$

that is (iii) is verified. □

Since J_ε has a mountain pass geometry, for any $\varepsilon \in (0, \varepsilon_0]$ we can define

$$c_\varepsilon = \inf_{\gamma \in \Gamma_\varepsilon} \max_{0 \leq t \leq 1} J_\varepsilon(\gamma(t)) \quad (2.18)$$

where

$$\Gamma_\varepsilon = \{\gamma \in C([0, 1], H_\varepsilon^s) : \gamma(0) = 0 \text{ and } J_\varepsilon(\gamma(1)) < 0\}. \quad (2.19)$$

By using Lemma 2.6, we are able to give the following estimate for c_ε :

Corollary 2.2. *There exist $m_1, m_2 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0]$*

$$m_1 \leq c_\varepsilon \leq m_2.$$

Proof. For any $\gamma \in \Gamma_\varepsilon$ we have

$$\gamma([0, 1]) \cap \{v \in H_\varepsilon^s : \|v\|_{H^s} = \rho\} \neq \emptyset.$$

Then, by using Lemma 2.6, we can deduce that

$$\max_{t \in [0, 1]} J_\varepsilon(\gamma(t)) \geq \inf_{\|v\|_{H^s} = \rho_0} J_\varepsilon(v) \geq \delta_0.$$

Now, we set $\gamma_0(t) = tv_0$, where $v_0 \in C_0^\infty(\mathbb{R}^N)$ is given in Lemma 2.6. Hence, we can see that

$$\begin{aligned} c_\varepsilon &= \inf_{\gamma \in \Gamma_\varepsilon} \left(\max_{t \in [0,1]} J_\varepsilon(\gamma(t)) \right) \\ &\leq \max_{t \in [0,1]} J_\varepsilon(\gamma_0(t)) \\ &\leq \sup_{\varepsilon \in (0, \varepsilon_0]} \left(\max_{t \in [0,1]} J_\varepsilon(\gamma_0(t)) \right) \\ &= m_2. \end{aligned}$$

Thus, we can put $m_1 = \delta_0$ and $m_2 = \sup_{\varepsilon \in (0, \varepsilon_0]} \left(\max_{t \in [0,1]} J_\varepsilon(\gamma_0(t)) \right)$.

□

In what follows, we investigate the boundedness of Cerami sequences corresponding to the mountain pass values c_ε . We recall that the existence of a Cerami sequence for J_ε follows by the following variant version of the mountain pass theorem.

Theorem 2.3. [25] *Let X be a real Banach space with its dual X^* , and suppose that $I \in C^1(X, \mathbb{R})$ satisfies*

$$\max\{I(0), I(e)\} \leq \mu < \alpha \leq \inf_{\|x\|=\rho} I(x),$$

for some $\mu < \alpha$, $\rho > 0$ and $e \in X$ with $\|e\| > \rho$. Let $c \geq \alpha$ be characterized by

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e\}$$

is the set of continuous paths joining 0 and e . Then, there exists a Cerami sequence $\{x_j\} \subset X$ at the level c that is

$$I(x_j) \rightarrow c \text{ and } (1 + \|x_j\|)\|I'(x_j)\|_* \rightarrow 0$$

as $j \rightarrow \infty$.

By using Lemma 2.6 and Theorem 2.3, we can deduce that for all $\varepsilon \in (0, \varepsilon_0]$ there exists a Cerami-sequence $(v_j) \subset H_\varepsilon^s$ such that

$$\begin{aligned} J_\varepsilon(v_j) &\rightarrow b_\varepsilon \\ (1 + \|v_j\|_{H_\varepsilon^s})\|J'_\varepsilon(v_j)\|_{H_\varepsilon^{-s}} &\rightarrow 0 \text{ as } j \rightarrow +\infty. \end{aligned}$$

The next result states that every critical point v_ε of J_ε at the level c_ε is uniformly bounded with respect to ε , that is

$$\limsup_{\varepsilon \rightarrow 0} \|v_\varepsilon\|_{H_\varepsilon^s} < +\infty. \quad (2.20)$$

Lemma 2.7. *Assume that f satisfies (f1)-(f3) and either (f4) or (f5). Suppose that there exists a sequence $(v_\varepsilon)_{\varepsilon \in (0, \varepsilon_1]}$ such that*

$$\begin{aligned} v_\varepsilon &\in H_\varepsilon^s, \\ J_\varepsilon(v_\varepsilon) &\in [m_1, m_2] \quad \forall \varepsilon \in (0, \varepsilon_1], \end{aligned} \quad (2.21)$$

$$(1 + \|v_\varepsilon\|_{H_\varepsilon^s})\|J'_\varepsilon(v_\varepsilon)\|_{H_\varepsilon^{-s}} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0 \quad (2.22)$$

with $0 < m_1 < m_2$. Then

$$\limsup_{\varepsilon \rightarrow 0} \|v_\varepsilon\|_{H_\varepsilon^s} < +\infty$$

Proof. Assume that (f4) holds, and let (v_ε) a sequence satisfying the above properties. Then

$$J_\varepsilon(v_\varepsilon) = \frac{1}{2}\|v_\varepsilon\|_{H_\varepsilon^s}^2 - \int_{\mathbb{R}^N} (1 - \chi(\varepsilon x)) \underline{F}(v_\varepsilon) + \chi(\varepsilon x) F(v_\varepsilon) dx \leq m_2. \quad (2.23)$$

Moreover, for any ε sufficiently small, we have

$$|J'_\varepsilon(v_\varepsilon)v_\varepsilon| \leq \|J'_\varepsilon(v_\varepsilon)\|_{H_\varepsilon^{-s}} \|v_\varepsilon\|_{H_\varepsilon^s} \leq \|J'_\varepsilon(v_\varepsilon)\|_{H_\varepsilon^{-s}} (1 + \|v_\varepsilon\|_{H_\varepsilon^s}) \leq 1,$$

that is

$$|J'_\varepsilon(v_\varepsilon)v_\varepsilon| = \left| \|v_\varepsilon\|_{H_\varepsilon^s}^2 - \int_{\mathbb{R}^N} (1 - \chi(\varepsilon x)) \underline{f}(v_\varepsilon)v_\varepsilon + \chi(\varepsilon x) f(v_\varepsilon)v_\varepsilon dx \right| \leq 1. \quad (2.24)$$

Taking into account the following facts

$$\begin{aligned} \frac{1}{2}\|v_\varepsilon\|_{H_\varepsilon^s}^2 &\leq \int_{\mathbb{R}^N} (1 - \chi_\varepsilon(x)) \underline{F}(v_\varepsilon) + \chi_\varepsilon(x) F(v_\varepsilon) dx + m_2, \\ -\frac{1}{\mu} &\leq \frac{1}{\mu} \left(-\|v\|_{H_\varepsilon^s}^2 + \int_{\mathbb{R}^N} (1 - \chi_\varepsilon(x)) \underline{f}(v_\varepsilon)v_\varepsilon + \chi_\varepsilon(x) f(v_\varepsilon)v_\varepsilon dx \right) \leq \frac{1}{\mu}, \end{aligned}$$

and

$$\chi(\varepsilon y) \left[F(v_\varepsilon) - \frac{1}{\mu} f(v_\varepsilon)v_\varepsilon \right] \leq 0$$

we have

$$\left(\frac{1}{2} - \frac{1}{\mu} \right) \|v_\varepsilon\|_{H_\varepsilon^s}^2 \leq \int_{\mathbb{R}^N} (1 - \chi(\varepsilon x)) \left(\underline{F}(v_\varepsilon) - \frac{1}{\mu} f(v_\varepsilon)v_\varepsilon \right) dx + m_2 + \frac{1}{\mu}.$$

By Lemma 2.4-(iii) we know that $tf(t) \geq 0$ for all $t \in \mathbb{R}$, so we get

$$\left(\frac{1}{2} - \frac{1}{\mu} \right) \|v_\varepsilon\|_{H_\varepsilon^s}^2 \leq \int_{\mathbb{R}^N} (1 - \chi(\varepsilon x)) \underline{F}(v_\varepsilon) dx + m_2 + \frac{1}{\mu}. \quad (2.25)$$

On the other hand, by Lemma 2.5-(iii), we can see that $\underline{f}(t) \leq \nu t$ and $\underline{F}(t) \leq F(t)$, so

$$\begin{aligned} \underline{F}(t) &\leq \int_0^t \nu \tau d\tau = \nu \frac{t^2}{2} \quad \text{for all } t \geq 0 \\ \underline{F}(t) &= 0 \quad \text{for all } t \leq 0 \end{aligned}$$

which give

$$\underline{F}(t) \leq \frac{\nu t^2}{2} \quad \text{for all } t \in \mathbb{R}.$$

Then

$$\begin{aligned} \int_{\mathbb{R}^N} (1 - \chi(\varepsilon x)) \underline{F}(v_\varepsilon) dx &\leq \frac{1}{2} \nu \|v_\varepsilon\|_{L^2(\mathbb{R}^N)}^2 \\ &\leq \frac{1}{2} \nu \left[\|v_\varepsilon\|_{L^2(\mathbb{R}^N)}^2 + \frac{1}{V_0} \|(-\Delta)^{\frac{s}{2}} v_\varepsilon\|_{L^2(\mathbb{R}^N)}^2 \right]. \end{aligned}$$

This, together with (2.25) yield

$$\left(\frac{1}{2} - \frac{1}{\mu} \right) \|v_\varepsilon\|_{H_\varepsilon^s}^2 \leq \frac{\nu}{2V_0} \|v_\varepsilon\|_{H_\varepsilon^s}^2 + m_2 + \frac{1}{\mu}$$

that is

$$\left[\left(\frac{1}{2} - \frac{1}{\mu} \right) - \frac{\nu}{2V_0} \right] \|v_\varepsilon\|_{H_\varepsilon^s}^2 \leq m_2 + \frac{1}{\mu}.$$

Taking into account (2.12) we get

$$\|v_\varepsilon\|_{H_\varepsilon^s}^2 \leq \frac{m_2 + \frac{1}{\mu}}{\left[\left(\frac{1}{2} - \frac{1}{\mu} \right) - \frac{\nu}{2V_0} \right]},$$

that is $\|v_\varepsilon\|_{H_\varepsilon^s}$ is bounded if ε is small enough.

Now, let us assume that (f5) holds. Arguing by contradiction, we assume that

$$\limsup_{\varepsilon \rightarrow 0} \|v_\varepsilon\|_{H_\varepsilon^s} = \infty.$$

Let $\varepsilon_j \rightarrow 0$ be a subsequence such that $\|v_{\varepsilon_j}\|_{H_{\varepsilon_j}^s} \rightarrow \infty$. For simplicity, we denote ε_j still by ε .

Set

$$w_\varepsilon = \frac{v_\varepsilon}{\|v_\varepsilon\|_{H_\varepsilon^s}}.$$

Clearly

$$\|w_\varepsilon\|_{H^s(\mathbb{R}^N)} = \frac{\|v_\varepsilon\|_{H_\varepsilon^s}}{\|v_\varepsilon\|_{H_\varepsilon^s}} \leq \frac{\|v_\varepsilon\|_{H_\varepsilon^s}}{\|v_\varepsilon\|_{H_\varepsilon^s}} = 1.$$

Then, we can see that there exists $C_1 > 0$ independent of ε such that

$$\|\chi_\varepsilon w_\varepsilon\|_{H^s(\mathbb{R}^N)} \leq C_1. \quad (2.26)$$

In fact, by using the facts $0 \leq \chi \leq 1$, $(|a| + |b|)^2 \leq 2a^2 + 2|b|^2$, $\varepsilon \in (0, \varepsilon_1]$, $s \in (0, 1)$ and the change of variable for radial functions, we get

$$\begin{aligned} & \iint_{\mathbb{R}^{2N}} \frac{|\chi(\varepsilon x)w_\varepsilon(x) - \chi(\varepsilon y)w_\varepsilon(y)|^2}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} V_0(\chi_\varepsilon w_\varepsilon)^2 dx \\ & \leq 2 \iint_{\mathbb{R}^{2N}} \frac{|\chi(\varepsilon x) - \chi(\varepsilon y)|^2}{|x - y|^{N+2s}} w_\varepsilon^2(x) dx dy + 2 \iint_{\mathbb{R}^N} \frac{|w_\varepsilon(x) - w_\varepsilon(y)|^2}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} V_0 w_\varepsilon^2 dx \\ & \leq 2\varepsilon^2 \|\nabla \chi\|_{L^\infty(\mathbb{R}^N)}^2 \int_{\mathbb{R}^N} w_\varepsilon^2(x) dx \int_{|z| \leq 1} \frac{1}{|z|^{N+2s-2}} dz + 8 \int_{\mathbb{R}^N} w_\varepsilon^2(x) dx \int_{|z| > 1} \frac{1}{|z|^{N+2s}} dz \\ & + 4 \iint_{\mathbb{R}^N} \frac{|w_\varepsilon(x) - w_\varepsilon(y)|^2}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} V_0 w_\varepsilon^2 dx \\ & \leq (2\varepsilon_1^2 \|\nabla \chi\|_{L^\infty(\mathbb{R}^N)}^2 \alpha_{N-1} + 8\alpha_{N-1} + V_0) \|w_\varepsilon\|_{L^2(\mathbb{R}^N)}^2 + 4[w_\varepsilon]^2 \leq C_1 \|w_\varepsilon\|_{H^s(\mathbb{R}^N)}^2 \leq C_1. \end{aligned}$$

Now, by using (2.22) we deduce that $\langle J'_\varepsilon(v_\varepsilon), \varphi \rangle = o(1)$ for any $\varphi \in H_\varepsilon^s$ that is

$$\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} v_\varepsilon (-\Delta)^{\frac{s}{2}} \varphi + V(\varepsilon x) v_\varepsilon \varphi dx = \int_{\mathbb{R}^N} [\chi_\varepsilon f(v_\varepsilon) \varphi + (1 - \chi_\varepsilon) \underline{f}(v_\varepsilon)] \varphi dx + o(1)$$

which gives

$$\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} w_\varepsilon (-\Delta)^{\frac{s}{2}} \varphi + V(\varepsilon x) w_\varepsilon \varphi dx = \int_{\mathbb{R}^N} \left[\chi_\varepsilon \frac{f(v_\varepsilon)}{v_\varepsilon} w_\varepsilon + (1 - \chi_\varepsilon) \frac{\underline{f}(v_\varepsilon)}{v_\varepsilon} w_\varepsilon \right] \varphi dx + o(1). \quad (2.27)$$

Taking $\varphi = w_\varepsilon^-$ in (2.27) and recalling that $f(t) = 0$ for all $t \leq 0$, we have

$$- \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} w_\varepsilon^-|^s + V(\varepsilon x) (w_\varepsilon^-)^2 dx = \int_{\mathbb{R}^N} \left[\chi_\varepsilon \frac{f(v_\varepsilon)}{v_\varepsilon} w_\varepsilon + (1 - \chi_\varepsilon) \frac{\underline{f}(v_\varepsilon)}{v_\varepsilon} w_\varepsilon \right] w_\varepsilon^- dx + o(1) = o(1),$$

so we get

$$\|w_\varepsilon^-\|_{H_\varepsilon}^2 \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \quad (2.28)$$

Now, we can observe that one of the following two cases must occur

Case 1: $\limsup_{\varepsilon \rightarrow 0} \left(\sup_{z \in \mathbb{R}^N} \int_{B_1(z)} |\chi_\varepsilon(x) w_\varepsilon|^2 dx \right) > 0$;

Case 2: $\limsup_{\varepsilon \rightarrow 0} \left(\sup_{z \in \mathbb{R}^N} \int_{B_1(z)} |\chi_\varepsilon(x) w_\varepsilon|^2 dx \right) = 0$.

Step1: Case 1 cannot occur under the assumption (f5) with $a = +\infty$.

We argue by contradiction, and we suppose that Case 1 occurs. Then, up to a subsequence, there exists $(x_\varepsilon) \subset \mathbb{R}^N$, $d > 0$ and $x_0 \in \overline{\Lambda}$ such that

$$\int_{B_1(x_\varepsilon)} |\chi_\varepsilon w_\varepsilon|^2 dx \rightarrow d > 0 \quad (2.29)$$

$$\varepsilon x_\varepsilon \rightarrow x_0 \in \overline{\Lambda}. \quad (2.30)$$

Moreover, in view of (2.29), it must be $B_1(x_\varepsilon) \cap \text{supp}(\chi_\varepsilon) \neq \emptyset$, so there exists $z_\varepsilon \in \text{supp}(\chi_\varepsilon)$ such that $\chi(\varepsilon z_\varepsilon) \neq 0$ and $|z_\varepsilon - x_\varepsilon| < 1$. Hence $|\varepsilon x_\varepsilon - \varepsilon z_\varepsilon| < \varepsilon$ yields $\varepsilon x_\varepsilon \in N_\varepsilon(\Lambda) = \{z \in \mathbb{R}^N : \text{dist}(z, \Lambda) < \varepsilon\}$, and we may assume that

$$\varepsilon x_\varepsilon \rightarrow x_0 \in \overline{\Lambda}. \quad (2.31)$$

Since $\|w_\varepsilon\|_{H^s(\mathbb{R}^N)} \leq 1$, we may suppose that

$$w_\varepsilon(\cdot + x_\varepsilon) \rightharpoonup w_0 \text{ in } H^s(\mathbb{R}^N). \quad (2.32)$$

Taking into account (2.31) and (2.32) we have

$$(\chi_\varepsilon w_\varepsilon)(\cdot + x_\varepsilon) \rightharpoonup \chi(x_0)w_0 \text{ in } H^s(\mathbb{R}^N).$$

Let us show that $\chi(x_0) \neq 0$ and $w_0 \geq 0$ ($\neq 0$). If by contradiction $\chi(x_0) = 0$, by the dominated convergence theorem and (2.29), we can see that

$$\begin{aligned} 0 < d &= \lim_{\varepsilon_j \rightarrow 0} \int_{B_1(x_{\varepsilon_j})} |\chi_{\varepsilon_j} w_{\varepsilon_j}|^2 dx \\ &= \lim_{\varepsilon_j \rightarrow 0} \int_{B_1(0)} |\chi_{\varepsilon_j} w_{\varepsilon_j}|^2 (x + x_{\varepsilon_j}) dx \\ &= \int_{B_1(0)} |\chi(x_0)w_0(x)|^2 dx = 0 \end{aligned}$$

that is a contradiction. For the same reason $w_0 \not\equiv 0$. By using (2.28) and (2.32), we can see that $w_0 \geq 0$.

Thus, there exists a set $K \subset \mathbb{R}^N$ such that

$$|K| > 0 \quad (2.33)$$

$$w_\varepsilon(x + x_\varepsilon) \rightarrow w_0(x) > 0 \quad \forall x \in K. \quad (2.34)$$

Taking $\varphi = w_\varepsilon$ in (2.27), we get

$$1 = \|w_\varepsilon\|_{H_\varepsilon}^2 = \int_{\mathbb{R}^N} \chi_\varepsilon \frac{f(v_\varepsilon)}{v_\varepsilon} w_\varepsilon^2 + (1 - \chi_\varepsilon) \frac{f(v_\varepsilon)}{v_\varepsilon} w_\varepsilon^2 dy + o(1),$$

and by using (iv) of Lemma 2.5, we deduce that

$$\limsup_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N} \chi_\varepsilon \frac{f(v_\varepsilon)}{v_\varepsilon} w_\varepsilon^2 dx \leq 1 \quad (2.35)$$

that is

$$\limsup_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N} \chi(\varepsilon x + \varepsilon x_\varepsilon) \frac{f(v_\varepsilon(x + x_\varepsilon))}{v_\varepsilon(x + x_\varepsilon)} w_\varepsilon(x + x_\varepsilon)^2 dx \leq 1.$$

By using (2.33), (2.34) and the definition of w_ε , we obtain

$$v_\varepsilon(x + x_\varepsilon) = w_\varepsilon(x + x_\varepsilon) \|v_\varepsilon(\cdot + x_\varepsilon)\|_{H_\varepsilon^s} \rightarrow w_0(x)(+\infty) = +\infty \quad \forall x \in K.$$

This, together with $\lim_{\xi \rightarrow \infty} \frac{f(\xi)}{\xi} = a = +\infty$ and Fatou's Lemma, yield

$$\begin{aligned} & \liminf_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N} \chi_\varepsilon(x + x_\varepsilon) \frac{f(v_\varepsilon(x + x_\varepsilon))}{w_\varepsilon(x + x_\varepsilon)} w_\varepsilon(x + x_\varepsilon)^2 dx \\ & \geq \liminf_{\varepsilon \rightarrow 0} \int_K \chi_\varepsilon(x + x_\varepsilon) \frac{f(v_\varepsilon(x + x_\varepsilon))}{v_\varepsilon(x + x_\varepsilon)} w_\varepsilon(y + y_\varepsilon)^2 dx = +\infty \end{aligned}$$

which contradicts (2.35).

Step 2: Case 1 cannot take place under the assumption (f5) with $a < +\infty$.

As in Step 1, we extract a subsequence and we assume that (2.29), (2.30) and (2.32) hold with $\chi(x_0) \neq 0$ and $w_0 \geq 0 (\not\equiv 0)$. We aim to prove that w_0 is a weak solution to

$$(-\Delta)^s w_0 + V(x_0)w_0 = (\chi(x_0)a + (1 - \chi(x_0))\nu)w_0 \text{ in } \mathbb{R}^N. \quad (2.36)$$

This provides a contradiction since $(-\Delta)^s$ has no eigenvalues in $H^s(\mathbb{R}^N)$. Fix $\varphi \in C_0^\infty(\mathbb{R}^N)$. Taking into account (2.30), (2.32) and the continuity of V , we can see that

$$\begin{aligned} & \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} w_\varepsilon(x + x_\varepsilon) (-\Delta)^{\frac{s}{2}} \varphi(x) + V(\varepsilon x + \varepsilon x_\varepsilon) w_\varepsilon \varphi dx \\ & \rightarrow \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} w_0 (-\Delta)^{\frac{s}{2}} \varphi + V(x_0) w_0 \varphi dx \end{aligned} \quad (2.37)$$

Now, we show that

$$\int_{\mathbb{R}^N} \frac{g(\varepsilon y + \varepsilon y_\varepsilon, v_\varepsilon(y + y_\varepsilon))}{v_\varepsilon(x + y_\varepsilon)} w_\varepsilon \varphi \rightarrow (\chi(x_0)a + (1 - \chi(x_0))\nu) \int_{\mathbb{R}^N} w_0 \varphi. \quad (2.38)$$

Take $R > 1$ such that $\text{supp } \varphi \subset B_R$. Then, by using the fact that $H^s(\mathbb{R}^N)$ is compactly embedded into $L_{loc}^2(\mathbb{R}^N)$, we get $\|w_\varepsilon - w_0\|_{L^2(B_R)}^2 \rightarrow 0$. Hence, there exists $h \in L^2(B_R)$ such that

$$|w_\varepsilon| \leq h \text{ a.e. in } B_R.$$

Since $a < +\infty$, there exists $C > 0$ such that $|\frac{g(x,t)}{t}| \leq C$ for any $t > 0$. We recall that

$$\frac{g(x,t)}{t} \rightarrow \chi(x)a + (1 - \chi(x))\nu < +\infty \text{ as } t \rightarrow +\infty.$$

Then

$$\begin{aligned} \left| \frac{g(\varepsilon x + \varepsilon x_\varepsilon, v_\varepsilon(x + x_\varepsilon))}{v_\varepsilon(x + x_\varepsilon)} w_\varepsilon \varphi \right| & \leq C \|\varphi\|_\infty |w_\varepsilon(x)| \\ & \leq C \|\varphi\|_\infty h(x) \in L^1(B_R) \end{aligned} \quad (2.39)$$

and

$$\frac{g(\varepsilon x + \varepsilon x_\varepsilon, v_\varepsilon(x + x_\varepsilon))}{v_\varepsilon(x + x_\varepsilon)} w_\varepsilon \rightarrow [\chi(x_0)a + (1 - \chi(x_0))\nu] w_0(x) \text{ a.e. in } B_R. \quad (2.40)$$

In fact, if $w_0(x) = 0$, being $|\frac{g(x,t)}{t}| \leq C$ for all $t > 0$ and $w_\varepsilon \rightarrow w_0 = 0$ a.e. in B_R , we get

$$\left| \frac{g(\varepsilon x + \varepsilon x_\varepsilon, v_\varepsilon(x + x_\varepsilon))}{v_\varepsilon(x + x_\varepsilon)} w_\varepsilon \right| \leq C |w_\varepsilon| \rightarrow 0.$$

If $w_0(x) \neq 0$, then $v_\varepsilon(x + x_\varepsilon) = w_\varepsilon(x + x_\varepsilon) \|v_\varepsilon(x + x_\varepsilon)\|_{H_\varepsilon^s} \rightarrow \infty$ and by using $w_\varepsilon \rightarrow w_0$ a.e. in B_R we have

$$\frac{g(\varepsilon x + \varepsilon x_\varepsilon, v_\varepsilon(x + x_\varepsilon))}{v_\varepsilon(x + x_\varepsilon)} \rightarrow \chi(x_0)a + (1 - \chi(x_0))\nu.$$

Then (2.40) holds. Taking into account (2.39) and (2.40), we can infer that (2.38) is true in view of the dominated convergence theorem.

Step 3: Case 2 cannot take place.

Assume by contradiction that Case 2 occur. Since (2.26) holds, and

$$\lim_{\varepsilon \rightarrow 0} \sup_{z \in \mathbb{R}^N} \int_{B_1(z)} |\chi_\varepsilon w_\varepsilon|^2 dx = 0,$$

by Lemma 2.2 we deduce that $\|\chi_\varepsilon w_\varepsilon\|_{L^{p+1}(\mathbb{R}^N)} \rightarrow 0$.

Now, for any $L > 1$, we can see that

$$J_\varepsilon\left(\frac{L}{\|v_\varepsilon\|}v_\varepsilon\right) = \frac{1}{2}L^2 - \int_{\mathbb{R}^N} \chi_\varepsilon F(Lw_\varepsilon) + (1 - \chi_\varepsilon)\underline{F}(Lw_\varepsilon).$$

By (iii) of Lemma 2.5, we have

$$\int_{\mathbb{R}^N} (1 - \chi_\varepsilon)\underline{F}(Lw_\varepsilon) dx \leq \int_{\mathbb{R}^N} \frac{1}{2}\nu L^2 |w_\varepsilon|^2 dx \leq \int_{\mathbb{R}^N} \frac{V_0}{4} L^2 |w_\varepsilon|^2 dx \leq \frac{L^2}{4} \|w_\varepsilon\|_{H^s}^2 \leq \frac{L^2}{4}.$$

As a consequence,

$$J_\varepsilon(Lw_\varepsilon) \geq \frac{1}{4}L^2 - \int_{\mathbb{R}^N} \chi_\varepsilon F(Lw_\varepsilon) dx. \quad (2.41)$$

By using (2.11) and Hölder inequality, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} \chi_\varepsilon F(Lw_\varepsilon) dx &\leq \int_{\mathbb{R}^N} \left[\frac{\delta}{2} L^2 |w_\varepsilon|^2 + C_\delta \frac{|Lw_\varepsilon|^{p+1}}{p+1} \chi_\varepsilon(y) \right] dy \\ &\leq \delta L^2 \|w_\varepsilon\|_{L^2(\mathbb{R}^N)}^2 + C_\delta L^{p+1} \|w_\varepsilon\|_{L^{p+1}(\mathbb{R}^N)}^p \|\chi_\varepsilon w_\varepsilon\|_{L^{p+1}(\mathbb{R}^N)} \\ &\leq \frac{\delta L^2}{V_0^2} \|w_\varepsilon\|_{H_\varepsilon^s}^2 + o(1). \end{aligned} \quad (2.42)$$

Putting together (2.41) and (2.42), we have

$$J_\varepsilon(Lw_\varepsilon) \geq \frac{1}{4}L^2 - \frac{\delta L^2}{V_0^2} \|w_\varepsilon\|_{H_\varepsilon^s}^2 - o(1) \quad \forall \delta > 0,$$

and by the arbitrariness of $\delta > 0$, we get

$$\liminf_{\varepsilon \rightarrow 0} J_\varepsilon(Lw_\varepsilon) \geq \frac{1}{4}L^2.$$

Since $\|v_\varepsilon\|_{H_\varepsilon^s} \rightarrow \infty$, we can see that $\frac{L}{\|v_\varepsilon\|_{H_\varepsilon^s}} \in (0, 1)$ for ε sufficiently small, and we get

$$\max_{t \in [0,1]} J_\varepsilon(tv_\varepsilon) \geq J_\varepsilon\left(\frac{L}{\|v_\varepsilon\|}v_\varepsilon\right) \geq \frac{1}{4}L^2.$$

Take $L > 0$ sufficiently large in order to have $J_\varepsilon(v_\varepsilon) \leq m_2 < \frac{1}{4}L^2$.

Then, we can find $t_\varepsilon \in (0, 1)$ such that

$$J_\varepsilon(t_\varepsilon v_\varepsilon) = \max_{t \in [0,1]} J_\varepsilon(tv_\varepsilon).$$

Hence,

$$J_\varepsilon(t_\varepsilon v_\varepsilon) = \max_{t \in [0,1]} J_\varepsilon(tv_\varepsilon) \geq \frac{1}{4}L^2 \rightarrow \infty \text{ as } L \rightarrow +\infty$$

that is

$$J_\varepsilon(t_\varepsilon v_\varepsilon) \rightarrow \infty \text{ per } \varepsilon \rightarrow 0. \quad (2.43)$$

Now, by using $\langle J'_\varepsilon(t_\varepsilon v_\varepsilon), (t_\varepsilon v_\varepsilon) \rangle = 0$, (2.22) and Corollary 2.1-(iv), we can see that

$$\begin{aligned}
J_\varepsilon(t_\varepsilon v_\varepsilon) &= J_\varepsilon(t_\varepsilon v_\varepsilon) - \frac{1}{2} \langle J'_\varepsilon(t_\varepsilon v_\varepsilon), (t_\varepsilon v_\varepsilon) \rangle \\
&= \int_{\mathbb{R}^N} \hat{G}(\varepsilon y, t_\varepsilon v_\varepsilon) dx \\
&\leq D^{k_\nu} \int_{\mathbb{R}^N} \hat{G}(\varepsilon y, v_\varepsilon) dx \\
&= D^{k_\nu} (J_\varepsilon(v_\varepsilon) - \frac{1}{2} \langle J'_\varepsilon(v_\varepsilon), v_\varepsilon \rangle) \\
&\leq D^{k_\nu} m_2 + o(1)
\end{aligned} \tag{2.44}$$

which contradicts (2.43). Then, the Case 2 does not occur.

Step 4: Conclusion.

Putting together Step 1, Step 2 and Step 3, we can deduce that $\|v_\varepsilon\|_{H_\varepsilon^s}$ is bounded as $\varepsilon \rightarrow 0$. \square

In the next Lemma, we prove that every Cerami sequence $(v_j) \subset H_\varepsilon^s$ at level c_ε is bounded and admits a convergent subsequence in H_ε^s .

Lemma 2.8. *Assume that f verifies (f1)-(f3) and either (f4) or (f5). Then there exists $\varepsilon_1 \in (0, \varepsilon_0]$ such that for any $\varepsilon \in (0, \varepsilon_1]$, and for any $(v_j) \subset H_\varepsilon^s$ satisfying*

$$J_\varepsilon(v_j) \rightarrow c > 0, \tag{2.45}$$

$$(1 + \|v_j\|_{H_\varepsilon^s}) \|J'_\varepsilon(v_j)\|_{H_\varepsilon^{-s}} \rightarrow 0 \text{ as } j \rightarrow \infty \tag{2.46}$$

for some $c > 0$, we get

- (i) $\|v_j\|_{H_\varepsilon^s}$ is bounded as $j \rightarrow +\infty$;
- (ii) there exists (j_k) and $v_0 \in H_\varepsilon^s$ such that $v_{j_k} \rightarrow v_0$ strongly in H_ε^s .

Proof. The proof of (i) can be done in similar way to the proof of Lemma 2.7, after suitable modifications. More precisely, in Step 1 of Lemma 2.7, for a given sequence (v_j) , there exists $(x_j) \subset \mathbb{R}^N$ such that

$$\int_{B_1(x_j)} |\chi_\varepsilon w_j|^2 dx \rightarrow d > 0.$$

The sequence (x_j) verifies $\varepsilon x_j \in N_\varepsilon(\Lambda)$, and we may assume that $\varepsilon x_j \rightarrow x_0 \in \overline{N_\varepsilon(\Lambda)}$, where x_0 is such that $\chi(\varepsilon x + x_0) \neq 0$ in B_1 .

In Step2, we replace (2.36) by

$$(-\Delta)^s w_0 + V(\varepsilon x + x_0) w_0 = (\chi(\varepsilon x + x_0) a + (1 - \chi(\varepsilon x + x_0)) \nu) w_0 \text{ in } \mathbb{R}^N \tag{2.47}$$

where $w_0 \in H^s(\mathbb{R}^N)$ is nonnegative and not identically zero. Indeed, by using the maximum principle, we can see that $w_0 > 0$ in \mathbb{R}^N . Now, we set $\tilde{w}(x) = w_0(\frac{x-x_0}{\varepsilon})$. Then \tilde{w} satisfies

$$\varepsilon^{2s} (-\Delta)^s \tilde{w} + V(x) \tilde{w} = (\chi(x) a + (1 - \chi(x)) \nu) \tilde{w}. \tag{2.48}$$

We aim to prove that this is impossible for $\varepsilon > 0$ sufficiently small. By using the extension technique [15], we can see that $\tilde{W} := \text{Ext}(\tilde{w})$ is a solution to the following problem

$$\begin{cases} \varepsilon^{2s} \operatorname{div}(y^{1-2s} \nabla \tilde{W}) = 0 & \text{in } \mathbb{R}_+^{N+1} \\ \frac{\partial \tilde{W}}{\partial \nu^{1-2s}} = -V(x) \tilde{w} + (\chi(x) a + (1 - \chi(x)) \nu) \tilde{w} & \text{on } \partial \mathbb{R}_+^{N+1} \end{cases} \tag{2.49}$$

where we have used the notation $w(x) = W(x, 0)$.

Take $R > 0$ such that

$$\chi(x) = 1 \text{ e } V(x) < a \quad \forall x \in B_R.$$

Let us introduce the following notations

$$\begin{aligned} B_R^+ &= \{(x, y) \in \mathbb{R}_+^{N+1} : y > 0, |(x, y)| < R\}, \\ \Gamma_R^+ &= \{(x, y) \in \mathbb{R}_+^{N+1} : y \geq 0, |(x, y)| = R\}, \\ \Gamma_R^0 &= \{(x, 0) \in \partial\mathbb{R}_+^{N+1} : |x| < R\}, \end{aligned}$$

and

$$H_{0, \Gamma_R^+}^1(B_R^+) = \{V \in H^1(B_R^+, y^{1-2s}) : V \equiv 0 \text{ on } \Gamma_R^+\}.$$

Let

$$\mu_R := \inf \left\{ \iint_{B_R^+} y^{1-2s} |\nabla U|^2 dx dy : U \in H_{0, \Gamma_R^+}^1(B_R^+), \int_{\Gamma_R^0} u^2 dx = 1 \right\}.$$

By the compactness of the embedding $H_{0, \Gamma_R^+}^1(B_R^+) \Subset L^2(\Gamma_R^0)$, it is not difficult to see that such infimum is achieved by a function $U_R \in H_{\Gamma_R^+}^1(B_R^+) \setminus \{0\}$. Moreover, we may assume that $U_R \geq 0$. Then, U_R is a solution, not identically zero, of

$$\begin{cases} \operatorname{div}(y^{1-2s} \nabla U_R) = 0 & \text{in } B_R^+ \\ \frac{\partial U_R}{\partial \nu^{1-2s}} = \mu_R U_R & \text{on } \Gamma_R^0 \\ U_R = 0 & \text{on } \Gamma_R^+ \end{cases} \quad (2.50)$$

It follows from the strong maximum principle [14] that $U_R > 0$ on $B_R^+ \cup \Gamma_R^0$. Let us note $\mu_R \geq 0$ and μ_R is a nonincreasing function of R . Indeed, μ_R is decreasing in R . In fact, if by contradiction we assume that $R_1 < R_2$ and $\mu_{R_1} = \mu_{R_2}$, we can multiply the equation $\operatorname{div}(y^{1-2s} \nabla U_{R_1})$ by U_{R_2} , and after an integration by parts, we can use the equalities satisfied by U_{R_1} and U_{R_2} , and the assumption $\mu_{R_1} = \mu_{R_2}$, to deduce that

$$\int_{\Gamma_{R_1}^+} \frac{\partial U_{R_1}}{\partial \nu^{1-2s}} U_{R_2} d\sigma = 0.$$

This gives a contradiction, because of $U_{R_2} > 0$ and $\frac{\partial U_{R_1}}{\partial \nu^{1-2s}} < 0$ on $\Gamma_{R_1}^+$.

Now, we extend $U_R = 0$ in $\mathbb{R}_+^{N+1} \setminus B_R^+$, so that $U_R \in H^1(\mathbb{R}_+^{N+1}, y^{1-2s})$. Therefore,

$$\begin{aligned} \varepsilon^{2s} \mu_R \int_{\Gamma_R^0} u_R \tilde{w} dx &= \iint_{B_R^+} y^{1-2s} \varepsilon^{2s} \nabla \tilde{W} \nabla U_R dx dy \\ &= - \int_{\Gamma_R^0} (V(x) - a) \tilde{w} u_R dx \end{aligned}$$

that is

$$\int_{\Gamma_R^0} (V(x) - a + \varepsilon^{2s} \mu_R) \tilde{w} u_R dx = 0. \quad (2.51)$$

But this gives a contradiction because of $V(x) - a + \mu_R \varepsilon^{2s} < 0$ in Γ_R^0 for $\varepsilon > 0$ small, $u_R \tilde{w} > 0$ in Γ_R^0 and $\mu_R > 0$.

In order to verify (ii), we fix $\varepsilon \in (0, \varepsilon_1]$ e (v_j) satisfying (2.45) and (2.46). By using (i), we can see that v_j is bounded in H_ε^s . Up to a subsequence, we may assume that

$$v_j \rightharpoonup v_0 \text{ in } H_\varepsilon^s.$$

Our claim, is to prove that $v_j \rightarrow v_0$ in H_ε^s . To do this, it suffices to show that

$$\limsup_{R \rightarrow \infty} \limsup_{j \rightarrow \infty} \int_{|x| \geq R} |(-\Delta)^{\frac{s}{2}} v_j|^2 + V(\varepsilon x) v_j^2 dx = 0. \quad (2.52)$$

Let us assume that (2.52) is true, and we prove that (2.52) implies the strong convergence in H_ε^s . To do this, we will show that

$$\int_{\mathbb{R}^N} g(\varepsilon x, v_j) v_j \, dx = \int_{\mathbb{R}^N} g(\varepsilon x, v_0) v_0 \, dx + o(1)$$

Clearly, we have

$$\begin{aligned} & \int_{\mathbb{R}^N} [g(\varepsilon x, v_j) v_j - g(\varepsilon x, v_0) v_0] \, dx \\ &= \int_{B_R} [g(\varepsilon x, v_j) v_j - g(\varepsilon x, v_0) v_0] \, dx + \int_{B_R^c} [g(\varepsilon x, v_j) v_j - g(\varepsilon x, v_0) v_0] \, dx =: (I) + (II) \end{aligned}$$

Now, $v_j \rightharpoonup v_0$ in H_ε^s , so, by using Theorem 2.1 we get $v_j \rightarrow v_0$ in $L^q(B_R)$, for any $R > 0$ and $q \in [2, 2_s^*)$. Then, the continuity of $g(\varepsilon x, \cdot)$, and Lebesgue convergence theorem give $(I) \rightarrow 0$. On the other hand, by using Corollary 2.1 (iii), we have for all $\delta > 0$

$$\begin{aligned} \left| \int_{|x| \geq R} [g(\varepsilon x, v_j) v_j - g(\varepsilon x, v_0) v_0] \, dx \right| &\leq \int_{|x| \geq R} |g(\varepsilon x, v_j) v_j - g(\varepsilon x, v_0) v_0| \, dx \\ &\leq \frac{1}{V_0} \|\sqrt{V(\varepsilon \cdot)} v_j\|_{L^2(|x| \geq R)}^2 + C_1 C_p \|v_j\|_{L^{p+1}(|x| \geq R)}^{p+1} \\ &\quad + \frac{1}{V_0} \|\sqrt{V_0} v_0\|_{L^2(|x| \geq R)}^2 + C_1 C_p \|v_0\|_{L^{p+1}(|x| \geq R)}^{p+1}. \end{aligned}$$

By using (2.52), Sobolev inequality in Theorem 2.1, and the fact that for any $u \in L^q(\mathbb{R}^N)$ it results

$$\lim_{R \rightarrow \infty} \int_{|x| \geq R} |u|^q \, dx = 0,$$

we get

$$\limsup_{R \rightarrow \infty} \limsup_{j \rightarrow \infty} \|\sqrt{V(\varepsilon \cdot)} v_j\|_{L^2(|x| \geq R)} = 0 = \limsup_{R \rightarrow \infty} \|\sqrt{V_0} v_0\|_{L^2(|x| \geq R)}$$

and

$$\limsup_{R \rightarrow \infty} \limsup_{j \rightarrow \infty} \|v_j\|_{L^{p+1}(|x| \geq R)} = 0 = \limsup_{R \rightarrow \infty} \|v_0\|_{L^{p+1}(|x| \geq R)}.$$

Taking into account that H_ε^s is a Hilbert space, $v_j \rightharpoonup v_0$ in H_ε^s and $\|v_j\|_{H_\varepsilon^s} \rightarrow \|v_0\|_{H_\varepsilon^s}$, we obtain that $v_j \rightarrow v_0$ in H_ε^s .

Now, we prove that (2.52) holds. Let $\eta_R \in C^\infty(\mathbb{R}^N, \mathbb{R})$ be a cut-off function such that

$$\begin{aligned} \eta_R(x) &= 0 \text{ for } |x| \leq \frac{R}{2} \\ \eta_R(x) &= 1 \text{ for } |x| \geq R \\ 0 &\leq \eta_R(x) \leq 1 \quad \forall x \in \mathbb{R}^N \\ |\nabla \eta_R(x)| &\leq \frac{C}{R} \quad \forall x \in \mathbb{R}^N \end{aligned}$$

Take $R > 0$ such that $\frac{\Delta}{\varepsilon} \subset B_{\frac{R}{2}}(0)$. Since $(v_j \eta_R)$ is bounded in H_ε^s , we can see that $\langle J'_\varepsilon(v_j), \eta_R v_j \rangle = o_j(1)$. Hence, we get

$$\begin{aligned} \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} v_j (-\Delta)^{\frac{s}{2}} (v_j \eta_R) \, dx + \int_{\mathbb{R}^N} V(\varepsilon x) v_j^2 \eta_R \, dx &= \int_{\mathbb{R}^N} \underline{f}(v_j) v_j \eta_R \, dx + o_j(1) \\ &\leq \nu \int_{\mathbb{R}^N} v_j^2 \eta_R \, dx + o_j(1). \end{aligned}$$

By our choice of ν , we can see that there exists $\alpha \in (0, 1)$ such that

$$\iint_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} v_j (-\Delta)^{\frac{s}{2}} (v_j \eta_R) dx + \alpha \int_{\mathbb{R}^N} V(\varepsilon x) v_j^2 \eta_R dx \leq o_j(1). \quad (2.53)$$

Now, we observe that

$$\begin{aligned} & \iint_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} v_j (-\Delta)^{\frac{s}{2}} (v_j \eta_R) dx \\ &= \iint_{\mathbb{R}^{2N}} \frac{(v_j(x) - v_j(y))(v_j(x) \eta_R(x) - v_j(y) \eta_R(y))}{|x - y|^{N+2s}} dx dy \\ &= \iint_{\mathbb{R}^{2N}} \eta_R(x) \frac{|v_j(x) - v_j(y)|^2}{|x - y|^{N+2s}} dx dy + \iint_{\mathbb{R}^{2N}} \frac{(v_j(x) - v_j(y))(\eta_R(x) - \eta_R(y))}{|x - y|^{N+2s}} v_j(y) dx dy \\ &=: A_{R,j} + B_{R,j}. \end{aligned} \quad (2.54)$$

Clearly

$$A_{R,j} \geq \int_{|x| \geq R} \int_{\mathbb{R}^N} \frac{|v_j(x) - v_j(y)|^2}{|x - y|^{N+2s}} dx dy. \quad (2.55)$$

By using Lemma 2.1 and the fact that (v_j) is bounded in $H^s(\mathbb{R}^N)$, we get

$$\begin{aligned} & \limsup_{R \rightarrow \infty} \limsup_{j \rightarrow \infty} |B_{R,j}| \\ & \leq \limsup_{R \rightarrow \infty} \limsup_{j \rightarrow \infty} \left(\iint_{\mathbb{R}^{2N}} \frac{|v_j(x) - v_j(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}} \left(\iint_{\mathbb{R}^{2N}} \frac{|\eta_R(x) - \eta_R(y)|^2}{|x - y|^{N+2s}} v_j^2(y) dx dy \right)^{\frac{1}{2}} \\ & \leq C \limsup_{R \rightarrow \infty} \limsup_{j \rightarrow \infty} \left(\iint_{\mathbb{R}^{2N}} \frac{|\eta_R(x) - \eta_R(y)|^2}{|x - y|^{N+2s}} v_j^2(y) dx dy \right)^{\frac{1}{2}} = 0. \end{aligned} \quad (2.56)$$

Putting together (2.53)-(2.56), we deduce that (2.52) holds. \square

Taking into account Lemma 2.7 and Lemma 2.8, we can infer the following result:

Corollary 2.3. *There exists $\varepsilon_1 \in (0, \varepsilon_0]$ such that for any $\varepsilon \in (0, \varepsilon_1]$ there exists a critical point $v_\varepsilon \in H_\varepsilon^s$ of $J_\varepsilon(v)$ satisfying $J_\varepsilon(v_\varepsilon) = c_\varepsilon$, where $c_\varepsilon \in [m_1, m_2]$ is defined as in (2.18)-(2.19). Moreover there exists $M > 0$ independent of $\varepsilon \in (0, \varepsilon_1]$ such that $\|v_\varepsilon\|_{H_\varepsilon^s} \leq M$ for any $\varepsilon \in (0, \varepsilon_1]$.*

3. LIMIT EQUATIONS

In the next section, we will see that the sequence of critical point obtained in Corollary 2.3, converges in some sense, to a sum of translated critical points associated to certain autonomous functionals. As proved in [8], least energy solutions for this limit functionals, have a mountain pass characterization. This property, will be fundamental to prove Theorem 1.1. For this reason, in this section we collect some important results concerning with the autonomous functionals corresponded to limit equations set in the whole space.

Firstly, we introduce some notations and definitions which will be useful later. Fix $x_0 \in \mathbb{R}^N$, and we define the autonomous functional $\Phi_{x_0} : H^s(\mathbb{R}^N) \rightarrow \mathbb{R}$ by setting

$$\Phi_{x_0}(v) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v|^2 + V(x_0) v^2 dx - \int_{\mathbb{R}^N} G(x_0, v) dx.$$

Clearly, $\Phi_{x_0} \in C^1(H^s(\mathbb{R}^N), \mathbb{R})$, and critical points of Φ_{x_0} are weak solutions to the equation

$$(-\Delta)^s u + V(x_0) u = g(x_0, u) \text{ in } \mathbb{R}^N.$$

Since Φ_{x_0} is autonomous, we can work on the fractional space $H_r^s(\mathbb{R}^N)$.

For any $x_0 \in \mathbb{R}^N$ and $u, v \in H^s(\mathbb{R}^N)$, we use the following notations

$$\begin{aligned}\langle u, v \rangle_{H_\varepsilon} &= \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + V(\varepsilon x) uv \, dx \\ \langle u, v \rangle_{x_0} &= \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + V(x_0) uv \, dx \\ \|v\|_{x_0}^2 &= \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v|^2 + V(x_0) v^2 \, dx\end{aligned}$$

We denote by

$$\begin{aligned}H(x, t) &= -\frac{1}{2}V(x)t^2 + \chi(x)F(t) + (1 - \chi(x))\underline{F}(t) \\ \Omega &= \left\{x \in \mathbb{R}^N : \sup_{t>0} H(x, t) > 0\right\}\end{aligned}$$

We note that $\Omega \subset \Lambda$. Indeed, if by contradiction $x \in \Omega \setminus \Lambda$, then

$$H(x, t) = -\frac{1}{2}V(x)t^2 + \underline{F}(t) \quad \text{and} \quad \sup_{t>0} H(x, t) > 0.$$

Now

$$\underline{F}(t) = \int_0^t \underline{f}(\tau) d\tau \leq \nu \frac{t^2}{2}$$

so

$$H(x, t) \leq t^2 \left[-\frac{1}{2}V(x) + \frac{\nu}{2} \right] < 0 \quad \text{for all } t > 0$$

since $\nu < V(x)$ for all $x \in \mathbb{R}^N$. Hence $\sup_{t>0} H(x, t) \leq 0$, that is a contradiction.

We can also see that $0 \in \{x \in \Lambda' : V(x) = \inf_{y \in \Lambda} V(y)\}$ and

$$\{x \in \Lambda' : V(x) = \inf_{y \in \Lambda} V(y)\} \subset \Omega. \quad (3.1)$$

To do this, we set $A = \{x \in \Lambda' : V(x) = \inf_{y \in \Lambda} V(y)\}$. If $x \in A$ and $x \notin \Omega$ then $H(x, t) \leq 0$ for all $t > 0$, that is

$$-\frac{1}{2} \left(\inf_{x \in \Lambda} V(x) \right) t^2 + F(t) \leq 0 \quad \text{for all } t > 0.$$

If (f4) holds, then

$$\frac{F(t)}{t^2} \rightarrow +\infty \text{ as } t \rightarrow +\infty$$

which gives

$$\frac{1}{2} \left(\inf_{x \in \Lambda} V(x) \right) \geq \frac{F(t)}{t^2} \rightarrow +\infty \text{ as } t \rightarrow +\infty$$

and this is a contradiction because of $\inf_{\Lambda} V < \min_{\partial \Lambda} V < +\infty$.

Now, we assume that (f5) is true. Thus we have

$$\frac{F(t)}{t^2} \rightarrow \frac{a}{2} \text{ as } t \rightarrow +\infty,$$

If $a = \infty$, then we can argue as before to deduce a contradiction.

Let $a \in (0, +\infty)$. Hence

$$\frac{1}{2} \left(\inf_{x \in \Lambda} V(x) \right) \geq \frac{F(t)}{t^2} \rightarrow \frac{a}{2}$$

which is impossible in view of $\inf_{x \in \Lambda} V(x) < a$.

Finally, we can observe that $\Omega = \Lambda$ if (f4) or (f5) with $a = \infty$ is true. Assume that (f4) holds.

We know that $\Omega \subset \Lambda$. Fix $x \in \Lambda$ and we show that $x \in \Omega$. If by contradiction $x \notin \Omega$, then

$$H(x, t) \leq 0 \quad \forall t > 0,$$

which implies that

$$-\frac{1}{2}V(x) + \chi(x)\frac{F(t)}{t^2} + (1 - \chi(x))\frac{F(t)}{t^2} \leq 0 \quad \text{for all } t > 0.$$

On the other hand, (f4) yields

$$\lim_{t \rightarrow +\infty} \frac{F(t)}{t^2} = +\infty$$

which together with (iv) of Lemma 2.5 gives

$$\begin{aligned} 0 &\geq -\frac{1}{2}V(x) + \chi(x)\frac{F(t)}{t^2} + (1 - \chi(x))\frac{F(t)}{t^2} \\ &\geq -\frac{1}{2}V(x) + \chi(x)\frac{F(t)}{t^2} \rightarrow +\infty \end{aligned}$$

which is a contradiction. Then, $x \in \Omega$.

Now, we assume that (f5) holds with $a = +\infty$. Fix $x \in \Lambda \setminus \Omega$. Hence,

$$-\frac{1}{2}V(x) + \chi(x)\frac{F(t)}{t^2} + (1 - \chi(x))\frac{F(t)}{t^2} \leq 0 \quad \text{for all } t > 0.$$

By (f5) we know that

$$\lim_{t \rightarrow +\infty} F(t) = +\infty$$

so

$$\lim_{t \rightarrow +\infty} \frac{F(t)}{t^2} = \lim_{t \rightarrow +\infty} \frac{f(t)}{2t} = \frac{a}{2} = \infty$$

and by using the fact that $\underline{F}(t) \geq 0$ for all $t \in \mathbb{R}^+$, we get

$$\begin{aligned} 0 &\geq -\frac{1}{2}V(x) + \chi(x)\frac{F(t)}{t^2} + (1 - \chi(x))\frac{F(t)}{t^2} \\ &\geq -\frac{1}{2}V(x) + \chi(x)\frac{F(t)}{t^2} \rightarrow +\infty \end{aligned}$$

which gives a contradiction. Therefore, $x \in \Omega$.

Now, we state the following Jeanjean-Tanaka type result [32] proved in [8] (see Lemma 13 in [8]) related to the study of the autonomous problem

$$(-\Delta)^s u = h(u) \text{ in } \mathbb{R}^N, \tag{3.2}$$

where $h \in C^1(\mathbb{R}, \mathbb{R})$ is an odd function satisfying Berestycki-Lions type assumptions

$$(h1) \quad \lim_{\xi \rightarrow 0} \frac{h(\xi)}{\xi} < 0;$$

$$(h2) \quad \lim_{\xi \rightarrow \infty} \frac{h(\xi)}{|\xi|^{2_s^*-1}} = 0;$$

$$(h3) \quad \text{there exists } \bar{\xi} > 0 \text{ such that } H(\bar{\xi}) > 0.$$

We recall that the existence of a solution to (3.2) has been established in [16].

Lemma 3.1. [8] *Assume that $h \in C^1(\mathbb{R}, \mathbb{R})$ is an odd function satisfying the Berestycki-Lions type assumptions (h1)-(h3). Let $\tilde{I} : H^s(\mathbb{R}^N) \rightarrow \mathbb{R}$ the functional defined by*

$$\tilde{I}(u) = \int_{\mathbb{R}^N} \frac{1}{2} |(-\Delta)^{\frac{s}{2}} u|^2 - H(u) dx.$$

Then \tilde{I} has a mountain pass geometry, and $c = m$ where m is defined as

$$m = \inf\{\tilde{I}(u) : u \in H^s(\mathbb{R}^N) - \{0\} \text{ is a solution to (3.2)}\}, \tag{3.3}$$

and

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} \tilde{I}(\gamma(t))$$

$$\text{where } \Gamma = \{\gamma \in C([0, 1], H^s(\mathbb{R}^N)) : \gamma(0) = 0, \tilde{I}(\gamma(1)) < 0\}.$$

Moreover, for any least energy solution $\omega(x)$ of (3.2) there exists a path $\gamma \in \Gamma$ such that

$$\tilde{I}(\gamma(t)) \leq m = \tilde{I}(\omega) \quad \forall t \in [0, 1] \quad (3.4)$$

$$\omega \in \gamma([0, 1]). \quad (3.5)$$

Next, we give the proof of the following lemma which we will use in the next section to obtain a concentration-compactness type result.

Lemma 3.2. *Assume that f satisfies (f1)-(f3). Then we have*

- (i) $\Phi_{x_0}(v)$ has non-zero critical points if and only if $x_0 \in \Omega$;
- (ii) There exists $\delta_1 > 0$, independent of $x_0 \in \mathbb{R}^N$, such that $\|v\|_{x_0} \geq \delta_1$ for any non zero critical point v of Φ_{x_0} .

Proof. Firstly, we extend $f(\xi)$ to an odd function on \mathbb{R} . Let us consider the function

$$h(\xi) = -V(x_0)\xi + g(x_0, \xi).$$

Clearly h is odd. Now, we show that h satisfies the assumptions (h1)-(h3) if and only if $x_0 \in \Omega$. From (f2) and (f3), follows that (h1) and (h2) hold. Since $\Omega = \{x \in \mathbb{R}^N : \sup_{\xi > 0} H(x, \xi) > 0\}$, we can see that (h3) is true if and only if $x_0 \in \Omega$. Then, by applying Theorem 1.1 in [16], we can find a non-zero critical point of Φ_{x_0} , that is (i) is satisfied.

Now, let v be a non-zero critical point of Φ_{x_0} . Then

$$\Phi'_{x_0}(v)v = 0 \Rightarrow \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v|^2 + V(x_0)v^2 dx - \int_{\mathbb{R}^N} g(x_0, v)v dx = 0.$$

By using (i) of Corollary 2.1, we get

$$\|v\|_{H^s(\mathbb{R}^N)}^2 - \int_{\mathbb{R}^N} f(v)v dx \leq 0,$$

so by (2.11), it follows that for any $\delta > 0$

$$\begin{aligned} \|v\|_{H^s(\mathbb{R}^N)}^2 &\leq \delta \|v\|_{L^2(\mathbb{R}^N)}^2 + C_\delta \|v\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} \\ &\leq \frac{\delta}{V_0} \|v\|_{H^s(\mathbb{R}^N)}^2 + C_\delta C'_{p+1} \|v\|_{H^s(\mathbb{R}^N)}^{p+1}. \end{aligned}$$

Hence

$$\left(1 - \frac{\delta}{V_0}\right) \|v\|_{H^s(\mathbb{R}^N)}^2 \leq C_\delta C'_{p+1} \|v\|_{H^s(\mathbb{R}^N)}^{p+1}.$$

Then we can find $\delta_1 > 0$ such that $\|v\|_{H^s(\mathbb{R}^N)} \geq \delta_1$ for any $x_0 \in \mathbb{R}^N$, and for any non-zero critical point v . Since $\|v\|_{x_0} \geq \|v\|_{H^s(\mathbb{R}^N)}$, we can infer that (ii) is verified. \square

For any $x \in \mathbb{R}^N$, we set

$$m(x) := \begin{cases} \text{least energy level of } \Phi_x(v) & x \in \Omega \\ \infty & x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

By Lemma 3.1, we can see that $m(x)$ is equal to the mountain pass value for $\Phi_x(v)$ if $x \in \Omega$, that is

$$m(x) = \inf_{\gamma \in \Gamma} \left(\max_{t \in [0, 1]} \Phi_x(\gamma(t)) \right).$$

Now, we prove the following result

Lemma 3.3.

$$m(x_0) = \inf_{x \in \mathbb{R}^N} m(x) \text{ if and only if } x_0 \in \Lambda \text{ e } V(x_0) = \inf_{x \in \Lambda} V(x).$$

In particular, $m(0) = \inf_{x \in \mathbb{R}^N} m(x)$.

Proof. Fix $x_0 \in \Lambda$ such that $V(x_0) = \inf_{x \in \Lambda} V(x)$. We note that $x_0 \in \Lambda'$. In fact, if $x_0 \in \Lambda - \Lambda'$, then

$$V(x_0) \geq \inf_{x \in \Lambda - \Lambda'} V(x) > \inf_{x \in \Lambda} V(x)$$

which is impossible. Hence $x_0 \in \Lambda'$ and $\chi(x_0) = 1$. Moreover, $x_0 \in \Omega$ by (3.1). Now, by using the fact that $V(x) \geq V(x_0)$ in Λ and $G(x, \xi) \leq F(\xi)$ for any $(x, \xi) \in \mathbb{R}^N \times \mathbb{R}$, we get for any $\bar{x} \in \Omega$

$$\begin{aligned} \Phi_{\bar{x}}(v) &= \frac{1}{2} \|(-\Delta)^{\frac{s}{2}} v\|_{L^2(\mathbb{R}^N)}^2 + \frac{1}{2} V(\bar{x}) \|v\|_{L^2(\mathbb{R}^N)}^2 - \int_{\mathbb{R}^N} G(x, v) dx \\ &\geq \frac{1}{2} \|(-\Delta)^{\frac{s}{2}} v\|_{L^2(\mathbb{R}^N)}^2 + \frac{1}{2} V(x_0) \|v\|_{L^2(\mathbb{R}^N)}^2 - \int_{\mathbb{R}^N} F(v) dx \\ &= \Phi_{x_0}(v) \end{aligned}$$

for any $v \in H^s(\mathbb{R}^N)$. This implies that $m(x_0) \leq m(x)$ for all $x \in \mathbb{R}^N$, so we have

$$m(x_0) \leq \inf_{x \in \mathbb{R}^N} m(x) \leq m(x_0)$$

that is $m(x_0) = \inf_{x \in \mathbb{R}^N} m(x)$.

Now, we fix $x' \in \Lambda$ such that $V(x') < V(x_0)$. Take $\gamma \in \Gamma$ such that (3.4)-(3.5) hold with $\tilde{I}(v) = \Phi_{x'}(v)$. Then, we deduce that

$$m(x_0) \leq \max_{t \in [0,1]} \Phi_{x_0}(\gamma(t)) < \max_{t \in [0,1]} \Phi_{x'}(\gamma(t)) = m(x').$$

□

Finally, we show the continuity of $m(x)$.

Proposition 3.1. *The function $m(x) : \mathbb{R}^N \mapsto (-\infty, +\infty]$ is continuous in the following sense:*

$$\begin{aligned} m(x_j) &\rightarrow m(x_0) & \text{if } x_j \rightarrow x_0 \in \Omega \\ m(x_j) &\rightarrow \infty & \text{if } x_j \rightarrow x_0 \in \mathbb{R}^N - \Omega \end{aligned}$$

Proof. Firstly, we fix $x_0 \in \Omega$ and we take $(x_j) \subset \Omega$ such that $x_j \rightarrow x$. We aim to prove that is upper semicontinuous, that is $\limsup_{j \rightarrow \infty} m(x_j) \leq m(x_0)$. In order to prove it, we show that for any fixed $\gamma \in \Gamma$, the map

$$L_\gamma : x_0 \in \Omega \mapsto \max_{t \in [0,1]} \Phi_x(\gamma(t))$$

is continuous. For any $t \in [0, 1]$, we have

$$\begin{aligned} &\Phi_{x_j}(\gamma(t)) - \Phi_{x_0}(\gamma(t)) \\ &= \frac{1}{2} \int_{\mathbb{R}^N} [V(x_j) - V(x_0)] |\gamma(t)(y)|^2 dx + \int_{\mathbb{R}^N} [G(x_j, \gamma(t)(y)) - G(x_0, \gamma(t)(y))] dx \rightarrow 0. \end{aligned}$$

Then

$$\left| \max_{t \in [0,1]} \Phi_{x_j}(\gamma(t)) - \max_{t \in [0,1]} \Phi_{x_0}(\gamma(t)) \right| \leq \max_{t \in [0,1]} |\Phi_{x_j}(\gamma(t)) - \Phi_{x_0}(\gamma(t))| \rightarrow 0.$$

Hence, being $m(x_0) = \inf_{\gamma \in \Gamma} L_\gamma(x_0)$, we deduce that $m(x_0)$ is upper semicontinuous. Now, we show that $m(x_0)$ is lower semicontinuous. To do this, we prove that for any least energy solution $u_j(x)$ of $\Phi_{x_j}(v)$ we have

- (i) $\|u_j\|_{H^s(\mathbb{R}^N)}$ is bounded as $j \rightarrow \infty$;

(ii) after extracting a subsequence, u_j has a non-zero weak limit u_0 and

$$\liminf_{j \rightarrow \infty} \Phi_{x_j}(u_j) \geq \Phi_{x_0}(u_0).$$

Indeed, it is clear that one can see that u_0 is a non-zero critical point of $\Phi_{x_0}(v)$, and then we have

$$\liminf_{j \rightarrow \infty} m(x_j) = \liminf_{j \rightarrow \infty} \Phi_{x_j}(u_j) \geq \Phi_{x_0}(u_0) \geq m(x_0).$$

Assume that $u_j \in H_r^s(\mathbb{R}^N)$. Then we know that $u_j(x)$ satisfies the Pohozaev Identity [16]:

$$\frac{N-2s}{2} \|(-\Delta)^{\frac{s}{2}} u_j\|_{L^2(\mathbb{R}^N)}^2 = N \int_{\mathbb{R}^N} H(x_j, u_j(x)) dx. \quad (3.6)$$

Now, we divide the proof in several steps.

Step 1: There exists $m_0, m_1 > 0$ (independent of j) such that

$$m_0 \leq m(x_j) \leq m_1 \quad \forall j \in \mathbb{N}.$$

The existence of m_1 follows by the fact that $m(x)$ is upper semicontinuous. Concerning m_0 , we note that

$$\Phi_{x_j}(v) \geq \frac{1}{2} \|(-\Delta)^{\frac{s}{2}} v\|_{L^2(\mathbb{R}^N)}^2 + \frac{1}{2} V_0 \|v\|_{L^2(\mathbb{R}^N)}^2 - \int_{\mathbb{R}^N} F(v) dx.$$

Then, taking m_0 to be the mountain pass value of

$$v \mapsto \frac{1}{2} \|(-\Delta)^{\frac{s}{2}} v\|_{L^2(\mathbb{R}^N)}^2 + \frac{1}{2} V_0 \|v\|_{L^2(\mathbb{R}^N)}^2 - \int_{\mathbb{R}^N} F(v) dx$$

we get the thesis.

Step 2: $\frac{N}{s} m_0 \leq \|(-\Delta)^{\frac{s}{2}} u_j\|_{L^2(\mathbb{R}^N)}^2 \leq \frac{N}{s} m_1$ for any $j \in \mathbb{N}$.

By using (3.6), we can see that

$$\begin{aligned} m(x_j) &= \Phi_{x_j}(u_j) \\ &= \frac{1}{2} \|(-\Delta)^{\frac{s}{2}} u_j\|_{L^2(\mathbb{R}^N)}^2 - \int_{\mathbb{R}^N} H(x_j, u_j(y)) dy \\ &= \frac{s}{N} \|(-\Delta)^{\frac{s}{2}} u_j\|_{L^2(\mathbb{R}^N)}^2 \end{aligned}$$

and by using $m_0 \leq m(x_j) \leq m_1$ for all $j \in \mathbb{N}$, we deduce that $Nm_0 \leq s \|(-\Delta)^{\frac{s}{2}} u_j\|_{L^2(\mathbb{R}^N)}^2 \leq Nm_1$.

Step 3: Boundedness of $\|u_j\|_{L^2(\mathbb{R}^N)}$.

Assume by contradiction that $\|u_j\|_{L^2(\mathbb{R}^N)} \rightarrow \infty$, and we set

$$t_j := \frac{1}{\|u_j\|_{L^2(\mathbb{R}^N)}^{\frac{N}{2}}} \rightarrow 0$$

and

$$\tilde{u}_j(x) := u_j\left(\frac{x}{t_j}\right).$$

Then $\|\tilde{u}_j\|_{L^2(\mathbb{R}^N)} = 1$ and

$$\|(-\Delta)^{\frac{s}{2}} \tilde{u}_j\|_{L^2(\mathbb{R}^N)}^2 = t_j^{N-2s} \|(-\Delta)^{\frac{s}{2}} u_j\|_{L^2(\mathbb{R}^N)}^2. \quad (3.7)$$

We aim to prove that $\tilde{u}_j \rightharpoonup 0$ in $H^s(\mathbb{R}^N)$. Up to a subsequence, we may suppose that $\tilde{u}_j \rightharpoonup \tilde{u}_0$. Since u_j is a critical point of $\Phi_{x_j}(v)$, we have

$$t_j^2 (-\Delta)^s \tilde{u}_j + V(x_j) \tilde{u}_j = g(x_j, \tilde{u}_j) \text{ in } \mathbb{R}^N. \quad (3.8)$$

Taking the limit as $j \rightarrow \infty$, we get

$$V(x_0) \tilde{u}_0 = g(x_0, \tilde{u}_0) \text{ in } \mathbb{R}^N.$$

Since $\tilde{u}_0 \in H^s(\mathbb{R}^N)$ and $0 \in \mathbb{R}$ is an isolated solution of $V(x_0)\xi = g(x_0, \xi)$, we deduce that $\tilde{u}_0 \equiv 0$. Now, we observe that

$$t_j^2 \|(-\Delta)^{\frac{s}{2}} \tilde{u}_j\|_{L^2(\mathbb{R}^N)}^2 + V_0 \|\tilde{u}_j\|_{L^2(\mathbb{R}^N)}^2 \leq \int_{\mathbb{R}^N} g(x_j, \tilde{u}_j) \tilde{u}_j dx. \quad (3.9)$$

By using Theorem 2.2 and the fact that $\tilde{u}_j \rightharpoonup 0$ in $H_r^s(\mathbb{R}^N)$, we know that $\tilde{u}_j \rightarrow 0$ in $L^q(\mathbb{R}^N)$ for any $q \in (2, 2_s^*)$. Then, in view of Lemma 2.3, we can see that

$$\int_{\mathbb{R}^N} g(x_j, \tilde{u}_j) \tilde{u}_j dx \rightarrow \int_{\mathbb{R}^N} g(x_0, \tilde{u}_0) \tilde{u}_0 dx = 0. \quad (3.10)$$

Taking into account (3.9), $\|\tilde{u}_j\|_{L^2(\mathbb{R}^N)} = 1$, $t_j \rightarrow 0$, (3.10) and Step 2, we obtain $V_0 \leq 0$, that is a contradiction.

Step 4: After extracting a subsequence, u_j has a non zero weak limit u_0 .

By using Step 2 and Step 3, $\|u_j\|_{H^s(\mathbb{R}^N)}$ is bounded. Then, we may assume that

$$\begin{aligned} u_j &\rightharpoonup 0 \text{ in } H^s(\mathbb{R}^N) \\ u_j &\rightarrow 0 \text{ in } L_{loc}^{p+1}(\mathbb{R}^N). \end{aligned}$$

Taking into account $\langle \Phi'_{x_j}(u_j), u_j \rangle = 0$ and Step 2, we can deduce that

$$0 < \frac{N}{s} m_0 \leq \|(-\Delta)^{\frac{s}{2}} u_j\|_{L^2(\mathbb{R}^N)}^2 + V_{x_j} \|u_j\|_{L^2(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} g(x_j, u_j) u_j dx. \quad (3.11)$$

As before, by using Lemma 2.3, we can see that

$$\int_{\mathbb{R}^N} g(x_j, u_j) u_j dx \rightarrow 0,$$

which is incompatible with (3.11).

Step 5: $\liminf_{j \rightarrow \infty} \Phi_{x_j}(u_j) \geq \Phi_{x_0}(u_0)$.

Let us note that

$$\begin{aligned} \Phi_{x_j}(u_j) &= \frac{1}{2} \|(-\Delta)^{\frac{s}{2}} u_j\|_{L^2(\mathbb{R}^N)}^2 + \frac{1}{2} V(x_j) \|u_j\|_{L^2(\mathbb{R}^N)}^2 - \int_{\mathbb{R}^N} G(x_j, u_j) dx \\ &\geq \frac{1}{2} \|(-\Delta)^{\frac{s}{2}} u_j\|_{L^2(\mathbb{R}^N)}^2 + \frac{1}{2} V(x_0) \|u_j\|_{L^2(\mathbb{R}^N)}^2 - \int_{\mathbb{R}^N} G(x_j, u_j) dx \end{aligned}$$

and

$$\begin{aligned} \|u_0\|_{L^2(\mathbb{R}^N)}^2 &\leq \liminf_{j \rightarrow \infty} \|u_j\|_{L^2(\mathbb{R}^N)}^2 \\ \|(-\Delta)^{\frac{s}{2}} u_0\|_{L^2(\mathbb{R}^N)}^2 &\leq \liminf_{j \rightarrow \infty} \|(-\Delta)^{\frac{s}{2}} u_j\|_{L^2(\mathbb{R}^N)}^2 \end{aligned}$$

by the weak lower semicontinuity of $H^s(\mathbb{R}^N)$ norm. Now, we show that

$$\int_{\mathbb{R}^N} G(x_j, u_j) dx \rightarrow \int_{\mathbb{R}^N} G(x_0, u_0) dx \text{ per } j \rightarrow \infty. \quad (3.12)$$

Then

$$\begin{aligned} \int_{\mathbb{R}^N} [G(x_j, u_j) - G(x_0, u_0)] dy &= \chi(x_j) \int_{\mathbb{R}^N} F(u_j) dx + (1 - \chi(x_j)) \int_{\mathbb{R}^N} \underline{F}(u_j) dx \\ &\quad + \chi(x_0) \int_{\mathbb{R}^N} F(u_0) dx + (1 - \chi(x_0)) \int_{\mathbb{R}^N} \underline{F}(u_0) dx. \end{aligned}$$

By using Lemma 2.3 and the continuity of χ , we deduce that (3.12) holds. Therefore

$$\begin{aligned} \liminf_{j \rightarrow \infty} \Phi_{x_j}(u_j) &\geq \frac{1}{2} \|(-\Delta)^{\frac{s}{2}} u_0\|_{L^2(\mathbb{R}^N)}^2 + \frac{1}{2} V(x_0) \|u_0\|_{L^2(\mathbb{R}^N)}^2 - \int_{\mathbb{R}^N} G(x_0, u_0) dx \\ &= \Phi_{x_0}(u_0). \end{aligned}$$

Finally, we deal with the case $x_0 \notin \Omega$.

Step 6: Let $x_0 \notin \Omega$ and (x_j) such that $x_j \rightarrow x_0$. Then $m(x_j) \rightarrow \infty$.

We argue by contradiction, and we assume that $m(x_j) \rightarrow \infty$. Then, there exists a subsequence, which denote again by (x_j) , such that $m(x_j)$ stays bounded as $j \rightarrow \infty$. By using the arguments of Steps 1-5, we can find a non zero critical point of $\Phi_{x_0}(v)$, which contradicts (i) of Lemma 3.2. \square

4. CONCENTRATION-COMPACTNESS RESULT

This section is devoted to the study of the behavior as $\varepsilon \rightarrow 0$, of critical points (v_ε) obtained in Corollary 2.3, that is such that

$$v_\varepsilon \in H_\varepsilon^s \quad (4.1)$$

$$J_\varepsilon(v_\varepsilon) \rightarrow c \in \mathbb{R} \quad (4.2)$$

$$(1 + \|v_\varepsilon\|_{H_\varepsilon^s}) \|J'_\varepsilon(v_\varepsilon)\|_{H_\varepsilon^{-s}} \rightarrow 0 \quad (4.3)$$

$$\|v_\varepsilon\|_{H_\varepsilon^s} \leq m \quad (4.4)$$

where c and m are independent of ε .

We begin proving the following concentration-compactness type result.

Lemma 4.1. *Assume that f satisfies (f1)-(f3) and $(v_\varepsilon)_{\varepsilon \in (0, \varepsilon_1]}$ satisfies (4.1)-(4.4).*

Then, there exists a subsequence $\varepsilon_j \rightarrow 0$, $l \in \mathbb{N} \cup \{0\}$, sequences $(y_{\varepsilon_j}^k) \subset \mathbb{R}^N$, $x^k \in \Omega$, $\omega^k \in H^1(\mathbb{R}^N) \setminus \{0\}$ ($k = 1, \dots, l$) such that

$$|y_{\varepsilon_j}^k - y_{\varepsilon_j}^{k'}| \rightarrow \infty \text{ as } j \rightarrow \infty, \forall k \neq k' \quad (4.5)$$

$$\varepsilon_j y_{\varepsilon_j}^k \rightarrow x^k \in \Omega \text{ as } j \rightarrow \infty \quad (4.6)$$

$$\omega^k \neq 0 \text{ e } \Phi'_{x^k}(\omega^k) = 0 \quad (4.7)$$

$$\|v_{\varepsilon_j} - \psi_{\varepsilon_j} \left(\sum_{k=1}^l \omega^k(\cdot - y_{\varepsilon_j}^k) \right)\|_{H_{\varepsilon_j}^s} \rightarrow 0 \text{ as } j \rightarrow \infty \quad (4.8)$$

$$J_{\varepsilon_j}(v_{\varepsilon_j}) \rightarrow \sum_{k=1}^l \Phi_{x^k}(\omega^k) \quad (4.9)$$

where $\psi_\varepsilon(x) = \psi(\varepsilon x)$, and $\psi \in C_0^\infty(\mathbb{R}^N, \mathbb{R})$ is such that $\psi(x) = 1$ for $x \in \Lambda$, $0 \leq \psi \leq 1$ on \mathbb{R}^N .

When $l = 0$, we have $\|v_{\varepsilon_j}\|_{H_{\varepsilon_j}^s} \rightarrow 0$ and $J_{\varepsilon_j}(v_{\varepsilon_j}) \rightarrow 0$.

Remark 4.1. *Since $\sup \psi(x\varepsilon)V(x\varepsilon) < \infty$ and ψ_ε , we can see that for all $w \in H^s(\mathbb{R}^N)$ and $\varepsilon > 0$, $\psi_\varepsilon w \in H_\varepsilon^s$ and there exists a constant $C > 0$, independent of ε , such that*

$$\|\psi_\varepsilon w\|_{H_\varepsilon^s} \leq C \|w\|_{H^s(\mathbb{R}^N)}. \quad (4.10)$$

Remark 4.2. For any $\omega \in H^s(\mathbb{R}^N)$ and sequence $(y_\varepsilon) \subset \mathbb{R}^N$ such that $\varepsilon y_\varepsilon \rightarrow x_0 \in \Lambda$ we have

$$\begin{aligned} & \|\psi_\varepsilon \omega(\cdot - y_\varepsilon)\|_{H_\varepsilon^s}^2 \\ &= \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}(\psi(\varepsilon x + \varepsilon y_\varepsilon)\omega(x))|^2 + V(\varepsilon x + \varepsilon y_\varepsilon)\psi(\varepsilon x + \varepsilon y_\varepsilon)^2\omega(x)^2 dx \\ &\rightarrow \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}\omega|^2 + V(x_0)\omega^2 dx = \|\omega\|_{x_0}^2 \quad \text{as } \varepsilon \rightarrow 0. \end{aligned} \quad (4.11)$$

We first prove that

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}(\psi(\varepsilon x + \varepsilon y_\varepsilon)\omega(x))|^2 dx \rightarrow \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}\omega|^2 dx. \quad (4.12)$$

In fact

$$\begin{aligned} & \iint_{\mathbb{R}^{2N}} \frac{|\psi(\varepsilon x + \varepsilon y_\varepsilon)\omega(x) - \psi(\varepsilon y + \varepsilon y_\varepsilon)\omega(y)|^2}{|x - y|^{N+2s}} dx dy \\ &= \iint_{\mathbb{R}^{2N}} \frac{|\psi(\varepsilon x + \varepsilon y_\varepsilon) - \psi(\varepsilon y + \varepsilon y_\varepsilon)|^2}{|x - y|^{N+2s}} (\omega(x))^2 dx dy + \iint_{\mathbb{R}^{2N}} \frac{|\omega(x) - \omega(y)|^2}{|x - y|^{N+2s}} (\psi(\varepsilon y + \varepsilon y_\varepsilon))^2 dx dy \\ &+ \iint_{\mathbb{R}^{2N}} \frac{(\psi(\varepsilon x + \varepsilon y_\varepsilon) - \psi(\varepsilon y + \varepsilon y_\varepsilon))(\omega(x) - \omega(y))}{|x - y|^{N+2s}} \omega(x)\psi(\varepsilon y + \varepsilon y_\varepsilon) dx dy \\ &=: A_\varepsilon + B_\varepsilon + C_\varepsilon. \end{aligned}$$

Now, by Lebesgue convergence theorem and $\psi(\varepsilon \cdot + \varepsilon y_\varepsilon) \rightarrow 1$ a.e., we get $B_\varepsilon \rightarrow [\omega]^2$. On the other hand

$$\begin{aligned} A_\varepsilon &= \int_{\mathbb{R}^N} dx \int_{|x-y| \leq \frac{1}{\varepsilon}} \frac{|\psi(\varepsilon x + \varepsilon y_\varepsilon) - \psi(\varepsilon y + \varepsilon y_\varepsilon)|^2}{|x - y|^{N+2s}} (\omega(x))^2 dy \\ &+ \int_{\mathbb{R}^N} dx \int_{|x-y| > \frac{1}{\varepsilon}} \frac{|\psi(\varepsilon x + \varepsilon y_\varepsilon) - \psi(\varepsilon y + \varepsilon y_\varepsilon)|^2}{|x - y|^{N+2s}} (\omega(x))^2 dy \\ &\leq \varepsilon^2 |\nabla \psi|^2 \alpha_{N-1} \int_{\mathbb{R}^N} \omega^2 dx \int_{|z| \leq \frac{1}{\varepsilon}} \frac{1}{|z|^{2s-1}} dz + 4\alpha_{N-1} \int_{\mathbb{R}^N} \omega^2 dx \int_{|z| > \frac{1}{\varepsilon}} \frac{1}{|z|^{2s+1}} dz \\ &= \varepsilon^{2s} \alpha_{N-1} \left(\frac{|\nabla \psi|^2}{2-2s} + \frac{2}{s} \right) \int_{\mathbb{R}^N} \omega^2 dx \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0, \end{aligned} \quad (4.13)$$

and by using

$$|C_\varepsilon| \leq [\omega] A_\varepsilon \rightarrow 0,$$

we can deduce that (4.12) holds. Since it is clear that

$$\int_{\mathbb{R}^N} V(\varepsilon x + \varepsilon y_\varepsilon)\psi(\varepsilon x + \varepsilon y_\varepsilon)^2\omega(x)^2 dx \rightarrow \int_{\mathbb{R}^N} V(x_0)\omega^2 dx, \quad (4.14)$$

we deduce that (4.12) and (4.14) imply (4.11).

Proof. We divide the proof in several steps. In what follows, we write ε instead of ε_j .

Step 1: Up to subsequence, $v_\varepsilon \rightharpoonup v_0$ in $H^s(\mathbb{R}^N)$ and v_0 is a critical point of $\Phi_0(v)$.

By using (4.4) and (2.16), we can see that $\|v_\varepsilon\|_{H^s(\mathbb{R}^N)} \leq m$. Then (v_ε) is bounded in $H^s(\mathbb{R}^N)$, and we can suppose that $v_\varepsilon \rightharpoonup v_0$ in $H^s(\mathbb{R}^N)$.

Let us show that v_0 is a critical point of $\Phi_0(v)$, that is $\langle \Phi_0'(v_0), \varphi \rangle = 0$ for any $\varphi \in H^s(\mathbb{R}^N)$. Since $C_0^\infty(\mathbb{R}^N)$ is dense in $H^s(\mathbb{R}^N)$, it is enough to prove it for any $\varphi \in C_0^\infty(\mathbb{R}^N)$.

Fix $\varphi \in C_0^\infty(\mathbb{R}^N)$. From (4.3) follows that

$$\int_{\mathbb{R}^N} [(-\Delta)^{\frac{s}{2}}v_\varepsilon(-\Delta)^{\frac{s}{2}}\varphi + V(\varepsilon x)v_\varepsilon\varphi - g(\varepsilon, v_\varepsilon)\varphi] dx \rightarrow 0.$$

Now, we show that

$$\langle J'_\varepsilon(v_\varepsilon), \varphi \rangle = \langle v_\varepsilon, \varphi \rangle_{H_\varepsilon^s} - \int_{\mathbb{R}^N} g(\varepsilon x, v_\varepsilon) \varphi \, dx \rightarrow \langle v_0, \varphi \rangle_0 - \int_{\mathbb{R}^N} g(0, v_0) \varphi \, dx.$$

Let us note that

$$\begin{aligned} & \langle v_\varepsilon, \varphi \rangle_{H_\varepsilon^s} - \langle v_0, \varphi \rangle_0 \\ &= \int_{\mathbb{R}^N} ((-\Delta)^{\frac{s}{2}} v_\varepsilon - (-\Delta)^{\frac{s}{2}} v_0) \varphi \, dx + \int_{\mathbb{R}^N} [V(\varepsilon x) - V(0)] v_\varepsilon \varphi \, dx + V(0) \int_{\mathbb{R}^N} (v_\varepsilon - v_0) \varphi \, dx \\ &=: (I) + (II) + (III). \end{aligned}$$

Then $(I), (III) \rightarrow 0$ because of $v_\varepsilon \rightharpoonup v_0$ in $H^s(\mathbb{R}^N)$, and

$$\begin{aligned} |(II)| &\leq C \|V_\varepsilon - V(0)\|_{L^\infty(\text{supp } \varphi)} \|v_\varepsilon\|_{H^s} \|\varphi\|_{L^2(\mathbb{R}^N)} \\ &\leq C m M \|V_\varepsilon - V(0)\|_{L^\infty(\text{supp } \varphi)} \rightarrow 0. \end{aligned}$$

Now, we prove that

$$\int_{\mathbb{R}^N} g(\varepsilon x, v_\varepsilon) \varphi \, dx \rightarrow \int_{\mathbb{R}^N} g(0, v_0) \varphi \, dx.$$

Indeed

$$\begin{aligned} \int_{\mathbb{R}^N} [g(\varepsilon x, v_\varepsilon) - g(0, v_0)] \varphi \, dx &= \int_{\text{supp } \varphi} [g(\varepsilon x, v_\varepsilon) - g(0, v_0)] \varphi \, dx \\ &= \int_{\text{supp } \varphi} [g(\varepsilon x, v_\varepsilon) - g(0, v_\varepsilon)] \varphi \, dx + \int_{\text{supp } \varphi} [g(0, v_\varepsilon) - g(0, v_0)] \varphi \, dx \end{aligned}$$

and, observing that

$$\int_{\mathbb{R}^N} [\chi(\varepsilon x) - 1] f(v_\varepsilon) \varphi \, dx \rightarrow 0$$

and

$$\int_{\mathbb{R}^N} (f(v_\varepsilon) - f(v_0)) \varphi \, dx \rightarrow 0,$$

we get the thesis. Hence

$$\Phi'_0(v_0) \varphi = \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} v_0 (-\Delta)^{\frac{s}{2}} \varphi + V(0) v_0 \varphi - g(0, v_0) \varphi \, dx = 0.$$

If $v_0 \not\equiv 0$, we set $y_\varepsilon^1 = 0$ and $\omega^1 = v_0$.

Step 2: Suppose that there exist $n \in \mathbb{N} \cup \{0\}$, $(y_\varepsilon^k) \subset \mathbb{R}^N$, $x^k \in \Omega$, $\omega^k \in H^s(\mathbb{R}^N)$ ($k = 1, \dots, n$) such that (4.5), (4.6), (4.7) of Lemma 4.1 hold for $k = 1, \dots, n$ and

$$v_\varepsilon(\cdot + y_\varepsilon^k) \rightharpoonup \omega^k \text{ in } H^s(\mathbb{R}^N). \quad (4.15)$$

Moreover, we assume that

$$\sup_{y \in \mathbb{R}^N} \int_{B_1(y)} \left| v_\varepsilon - \psi_\varepsilon \sum_{k=1}^n \omega^k(x - y_\varepsilon^k) \right|^2 dx \rightarrow 0. \quad (4.16)$$

Then,

$$\left\| v_\varepsilon - \psi_\varepsilon \sum_{k=1}^n \omega^k(\cdot - y_\varepsilon^k) \right\|_{H_\varepsilon^s}^2 \rightarrow 0. \quad (4.17)$$

Set

$$\xi_\varepsilon(x) = v_\varepsilon(x) - \psi_\varepsilon(x) \sum_{k=1}^n \omega^k(x - y_\varepsilon^k).$$

From (4.10) follows that

$$\begin{aligned} \|\xi_\varepsilon\|_{H_\varepsilon^s} &\leq \|v_\varepsilon\|_{H_\varepsilon^s} + \|\psi_\varepsilon \sum_{k=1}^n \omega^k(\cdot - y_\varepsilon^k)\|_{H_\varepsilon^s} \\ &\leq m + C \sum_{k=1}^n \|\omega^k\|_{H^s(\mathbb{R}^N)} \end{aligned}$$

and being $\|\xi_\varepsilon\|_{H^s(\mathbb{R}^N)} \leq \|\xi_\varepsilon\|_{H_\varepsilon^s}$ we deduce that (ξ_ε) is bounded in $H^s(\mathbb{R}^N)$.

By (4.16) and Lemma 2.2, we have $\|\xi_\varepsilon\|_{L^{p+1}(\mathbb{R}^N)} \rightarrow 0$ as $\varepsilon \rightarrow 0$. Now, a direct calculation shows that

$$\begin{aligned} \|\xi_\varepsilon\|_{H_\varepsilon^s}^2 &= \langle v_\varepsilon - \psi_\varepsilon \sum_{k=1}^n \omega^k(\cdot - y_\varepsilon^k), \xi_\varepsilon \rangle_{H_\varepsilon} \\ &= \langle v_\varepsilon, \xi_\varepsilon \rangle_{H_\varepsilon} - \sum_{k=1}^n \langle \psi_\varepsilon \omega^k(\cdot - y_\varepsilon^k), \xi_\varepsilon \rangle_{H_\varepsilon} \end{aligned} \quad (4.18)$$

Our claim, is to prove that for $k = 1, \dots, n$

$$\langle \psi_\varepsilon \omega^k(\cdot - y_\varepsilon^k), \xi_\varepsilon \rangle_{H_\varepsilon^s} = \langle \omega^k(\cdot - y_\varepsilon^k), \psi_\varepsilon \xi_\varepsilon \rangle_{x^k} + o(1) \quad (4.19)$$

Indeed

$$\begin{aligned} &\langle \psi_\varepsilon \omega^k(y - y_\varepsilon^k), \xi_\varepsilon \rangle_{H_\varepsilon^s} - \langle \omega^k(y - y_\varepsilon^k), \psi_\varepsilon \xi_\varepsilon \rangle_{x^k} \\ &= \left[\iint_{\mathbb{R}^{2N}} \frac{(\psi_\varepsilon(x) - \psi_\varepsilon(y))(\xi_\varepsilon(x) - \xi_\varepsilon(y))\omega^k(x - y_\varepsilon^k)}{|x - y|^{N-2s}} dx dy \right. \\ &\quad \left. - \iint_{\mathbb{R}^{2N}} \frac{(\psi_\varepsilon(x) - \psi_\varepsilon(y))(\omega^k(x - y_\varepsilon^k) - \omega^k(y - y_\varepsilon^k))\xi_\varepsilon(x)}{|x - y|^{N-2s}} dx dy \right] \\ &\quad + \int_{\mathbb{R}^N} (V(\varepsilon x + \varepsilon y_\varepsilon^k) - V(x^k)) \psi(\varepsilon x + \varepsilon y_\varepsilon^k) \omega^k(x) \xi_\varepsilon(x + y_\varepsilon^k) dx \\ &=: (I) + (II) \end{aligned}$$

We note that

$$\begin{aligned} &\left| \iint_{\mathbb{R}^{2N}} \frac{(\psi_\varepsilon(x) - \psi_\varepsilon(y))(\xi_\varepsilon(x) - \xi_\varepsilon(y))\omega^k(x - y_\varepsilon^k)}{|x - y|^{N-2s}} dx dy \right| \\ &\leq \left(\iint_{\mathbb{R}^{2N}} \frac{|\xi_\varepsilon(x) - \xi_\varepsilon(y)|^2}{|x - y|^{N-2s}} dx dy \right)^{\frac{1}{2}} \left(\iint_{\mathbb{R}^{2N}} \frac{|\psi_\varepsilon(x) - \psi_\varepsilon(y)|^2 (\omega^k(x - y_\varepsilon^k))^2}{|x - y|^{N-2s}} dx dy \right)^{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} &\left| \iint_{\mathbb{R}^{2N}} \frac{(\psi_\varepsilon(x) - \psi_\varepsilon(y))(\omega^k(x - y_\varepsilon^k) - \omega^k(y - y_\varepsilon^k))\xi_\varepsilon(x)}{|x - y|^{N-2s}} dx dy \right| \\ &\leq \left(\iint_{\mathbb{R}^{2N}} \frac{|\omega^k(x - y_\varepsilon^k) - \omega^k(y - y_\varepsilon^k)|^2}{|x - y|^{N-2s}} dx dy \right)^{\frac{1}{2}} \left(\iint_{\mathbb{R}^{2N}} \xi_\varepsilon^2(x) \frac{|\psi_\varepsilon(x) - \psi_\varepsilon(y)|^2}{|x - y|^{N-2s}} dx dy \right)^{\frac{1}{2}} \end{aligned}$$

so, by using the fact that $\|\xi_\varepsilon\|_{H^s(\mathbb{R}^N)} \leq \bar{C}_1$ and $\|\omega^k\|_{H^s(\mathbb{R}^N)} \leq \bar{C}_2$, for some $\bar{C}_1, \bar{C}_2 > 0$, we can argue as in the proof of (4.13) (to prove that $A_\varepsilon \rightarrow 0$), to see that $(I) \rightarrow 0$.

By using (4.5) and (4.16), we can deduce that

$$\begin{aligned}\xi_\varepsilon(\cdot + y_\varepsilon^k) &\rightharpoonup 0 \text{ in } H^s(\mathbb{R}^N) \\ \xi_\varepsilon(\cdot + y_\varepsilon^k) &\rightarrow 0 \text{ in } L^2_{loc}(\mathbb{R}^N).\end{aligned}\tag{4.20}$$

Now

$$\begin{aligned}(II) &= \int_{B_R^c} (V(\varepsilon x + \varepsilon y_\varepsilon^k) - V(x^k)) \psi(\varepsilon x + \varepsilon y_\varepsilon^k) \omega^k(x) \xi_\varepsilon(x + y_\varepsilon^k) dx \\ &\quad + \int_{B_R} (V(\varepsilon x + \varepsilon y_\varepsilon^k) - V(x^k)) \psi(\varepsilon x + \varepsilon y_\varepsilon^k) \omega^k(x) \xi_\varepsilon(x + y_\varepsilon^k) dx.\end{aligned}$$

By (4.20), we can see that

$$\begin{aligned}&\left| \int_{B_R} (V(\varepsilon x + \varepsilon y_\varepsilon^k) - V(x^k)) \psi(\varepsilon x + \varepsilon y_\varepsilon^k) \omega^k(x) \xi_\varepsilon(x + y_\varepsilon^k) dx \right| \\ &\leq \| (V(\varepsilon \cdot + \varepsilon y_\varepsilon^k) - V(x^k)) \psi(\varepsilon \cdot + \varepsilon y_\varepsilon^k) \|_{L^\infty(\mathbb{R}^N)} \|\omega^k\|_{L^2(\mathbb{R}^N)} \|\xi_\varepsilon(\cdot + y_\varepsilon^k)\|_{L^2(B_R)} \\ &\leq C \|\xi_\varepsilon(\cdot + y_\varepsilon^k)\|_{L^2(B_R)} \rightarrow 0\end{aligned}$$

On the other hand

$$\begin{aligned}&\left| \int_{B_R^c} (V(\varepsilon x + \varepsilon y_\varepsilon^k) - V(x^k)) \psi(\varepsilon x + \varepsilon y_\varepsilon^k) \omega^k(x) \xi_\varepsilon(x + y_\varepsilon^k) dy \right| \\ &\leq \| (V(\varepsilon \cdot + \varepsilon y_\varepsilon^k) - V(x^k)) \psi(\varepsilon \cdot + \varepsilon y_\varepsilon^k) \|_{L^\infty(\mathbb{R}^N)} \|\omega^k\|_{L^2(B_R^c)} \|\xi_\varepsilon(\cdot + y_\varepsilon^k)\|_{L^2(\mathbb{R}^N)} \\ &\leq C' \|\omega^k\|_{L^2(B_R^c)} \rightarrow 0,\end{aligned}$$

so, we can infer that $(II) \rightarrow 0$.

Putting together (4.18) and (4.19), we find

$$\begin{aligned}\|\xi_\varepsilon\|_{H_\varepsilon^s}^2 &= \langle v_\varepsilon, \xi_\varepsilon \rangle_{H_\varepsilon^s} - \sum_{k=1}^n \langle \omega^k(\cdot - y_\varepsilon^k), \psi_\varepsilon \xi_\varepsilon \rangle_{x^k} + o(1) \\ &= J'_\varepsilon(v_\varepsilon) \xi_\varepsilon + \int_{\mathbb{R}^N} g(\varepsilon x, v_\varepsilon) \xi_\varepsilon dx - \sum_{k=1}^n \left(\langle \Phi'_{x^k}(\omega^k(\cdot - y_\varepsilon^k)), (\psi_\varepsilon \xi_\varepsilon) \rangle \right. \\ &\quad \left. + \int_{\mathbb{R}^N} g(x^k, \omega^k(x - y_\varepsilon^k)) \psi_\varepsilon \xi_\varepsilon dx \right) + o(1) \\ &= \int_{\mathbb{R}^N} g(\varepsilon x, v_\varepsilon) \xi_\varepsilon dx - \sum_{k=1}^n \int_{\mathbb{R}^N} g(x^k, \omega^k(x - y_\varepsilon^k)) \psi_\varepsilon \xi_\varepsilon dx + o(1) \\ &= (III) - \sum_{k=1}^n (IV) + o(1).\end{aligned}$$

By Corollary 2.1 (iii), we have

$$\begin{aligned}|(III)| &\leq \delta \int_{\mathbb{R}^N} |v_\varepsilon \xi_\varepsilon| dx + C_\delta \int_{\mathbb{R}^N} |v_\varepsilon|^p |\xi_\varepsilon| dx \\ &\leq \delta \|v_\varepsilon\|_{L^2(\mathbb{R}^N)} \|\xi_\varepsilon\|_{L^2(\mathbb{R}^N)} + C_\delta \|v_\varepsilon\|_{L^{p+1}(\mathbb{R}^N)}^p \|\xi_\varepsilon\|_{L^{p+1}(\mathbb{R}^N)}.\end{aligned}$$

and by using $\|\xi_\varepsilon\|_{L^{p+1}(\mathbb{R}^N)} \rightarrow 0$ as $\varepsilon \rightarrow 0$, and the boundedness of $\|v_\varepsilon\|_{L^2(\mathbb{R}^N)}$ and $\|\xi_\varepsilon\|_{L^2(\mathbb{R}^N)}$ we get $(III) \rightarrow 0$. In view of (4.20), we can see that $(IV) \rightarrow 0$. Hence, $\|\xi_\varepsilon\|_{H_\varepsilon^s} \rightarrow 0$, that is (4.17) holds.

Step 3: Suppose that there exist $n \in \mathbb{N} \cup \{0\}$, $(y_\varepsilon) \subset \mathbb{R}^N$, $x^k \in \Omega$, $\omega^k \in H^1(\mathbb{R}^N) - \{0\}$ such that (4.5), (4.6), (4.7) and (4.15) hold. Assume also that for any $\varepsilon > 0$ there exists $z_\varepsilon \in \mathbb{R}^N$ such that

$$\int_{B_1(z_\varepsilon)} \left| v_\varepsilon - \psi_\varepsilon \sum_{k=1}^n \omega^k(x - y_\varepsilon^k) \right|^2 dx \rightarrow c > 0. \quad (4.21)$$

Then there exists $x^{k+1} \in \Omega$ and $\omega^{k+1} \in H^s(\mathbb{R}^N) - \{0\}$ such that

$$|z_\varepsilon - y_\varepsilon^k| \rightarrow \infty \quad \forall k = 1, \dots, n \quad (4.22)$$

$$\varepsilon z_\varepsilon \rightarrow x^{k+1} \in \Omega \quad (4.23)$$

$$v_\varepsilon(\cdot + z_\varepsilon) \rightharpoonup \omega^{k+1} \neq 0 \text{ in } H^s(\mathbb{R}^N) \quad (4.24)$$

$$\Phi'_{x^{k+1}}(\omega^{k+1}) = 0 \quad (4.25)$$

It is easy to see that z_ε verifies (4.22). Indeed, if by contradiction $|z_\varepsilon - y_\varepsilon^k| \rightarrow z \neq \infty$ then there exists $C > 0$ such that

$$|z_\varepsilon - y_\varepsilon^k - z| \leq C,$$

so, by using (4.20), we get

$$\int_{B_1(z_\varepsilon - y_\varepsilon^k)} |\xi_\varepsilon(x + y_\varepsilon^k)|^2 dx \leq \int_{B_{C+1}(z)} |\xi_\varepsilon(x + y_\varepsilon^k)|^2 dx \rightarrow 0$$

which contradicts (4.21). Combining (4.21) and Lemma 2.2, it is easy to see that there exists $\omega^{k+1} \in H^s(\mathbb{R}^N) - \{0\}$ such that (4.24) holds true. In fact, if $\omega^{k+1} \equiv 0$, then

$$\begin{aligned} \int_{B_1(z_\varepsilon)} |v_\varepsilon(x)|^2 dx &= \int_{B_1(z_\varepsilon)} \left| v_\varepsilon(x) - \psi_\varepsilon(x) \sum_{k=1}^n \omega^k(x - y_\varepsilon^k) + \psi_\varepsilon(x) \sum_{k=1}^n \omega^k(x - y_\varepsilon^k) \right|^2 dx \\ &\geq \frac{1}{2} \int_{B_1(z_\varepsilon)} \left| v_\varepsilon(x) - \psi_\varepsilon(x) \sum_{k=1}^n \omega^k(x - y_\varepsilon^k) \right|^2 dx - \int_{B_1(z_\varepsilon)} \left| \psi_\varepsilon(x) \sum_{k=1}^n \omega^k(x - y_\varepsilon^k) \right|^2 dx \\ &= \frac{1}{2} \int_{B_1(z_\varepsilon)} |\xi_\varepsilon(y)|^2 dx - \int_{B_1(z_\varepsilon)} \psi_\varepsilon^2(x) \left| \sum_{k=1}^n \omega^k(x - y_\varepsilon^k) \right|^2 dx \end{aligned}$$

and

$$\begin{aligned} \int_{B_1(z_\varepsilon)} \psi_\varepsilon^2(x) \left| \sum_{k=1}^n \omega^k(x - y_\varepsilon^k) \right|^2 dx &\leq C \sum_{k=1}^n \int_{B_1(z_\varepsilon)} |\omega^k(x - y_\varepsilon^k)|^2 dx \\ &= C \sum_{k=1}^n \int_{B_1(0)} |\omega^k(x + z_\varepsilon - y_\varepsilon^k)|^2 dx \rightarrow 0 \end{aligned}$$

because of $|z_\varepsilon - y_\varepsilon^k| \rightarrow \infty$ and $\omega^k \in H^s(\mathbb{R}^N)$.

Hence

$$\int_{B_1(z_\varepsilon)} |v_\varepsilon(x)|^2 dx \rightarrow a \geq \frac{c}{2} > 0.$$

On the other hand, $v_\varepsilon(\cdot + z_\varepsilon) \rightharpoonup 0$ in $H^s(\mathbb{R}^N)$, so we can see that

$$\int_{B_1(z_\varepsilon)} |v_\varepsilon(x)|^2 dx = \int_{B_1(0)} |v_\varepsilon(x + z_\varepsilon)|^2 dx \rightarrow 0$$

that is a contradiction.

Now, we show (4.23). Firstly, we prove that $\limsup_{\varepsilon \rightarrow 0} |\varepsilon z_\varepsilon| < \infty$.

Assume by contradiction that $|\varepsilon z_\varepsilon| \rightarrow \infty$. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ be a cut-off function such that $\varphi \geq 0$, $\varphi(0) = 1$ and let $\varphi_R(x) = \varphi(x/R)$. Since $(\varphi_R(\cdot - z_\varepsilon)v_\varepsilon)$ is bounded in H_ε^s , we obtain

$$\langle J'_\varepsilon(v_\varepsilon), \varphi_R(\cdot - z_\varepsilon)v_\varepsilon \rangle \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0,$$

that is

$$\begin{aligned} & \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} v_\varepsilon(x + z_\varepsilon) (-\Delta)^{\frac{s}{2}} (\varphi_R(x) v_\varepsilon(x + z_\varepsilon)) + V(\varepsilon x + \varepsilon z_\varepsilon) v_\varepsilon(x + z_\varepsilon)^2 \varphi_R(x) dx \\ & - \int_{\mathbb{R}^N} g(\varepsilon x + \varepsilon z_\varepsilon, v_\varepsilon(x + z_\varepsilon)) v_\varepsilon(x + z_\varepsilon) \varphi_R(x) dx \rightarrow 0. \end{aligned} \quad (4.26)$$

Let us note that $|\varepsilon z_\varepsilon| \rightarrow \infty$, yields

$$g(\varepsilon x + \varepsilon z_\varepsilon, v_\varepsilon(x + z_\varepsilon)) = \underline{f}(v_\varepsilon(x + z_\varepsilon)) \text{ on } \text{supp } \varphi_R$$

for any ε small enough. Moreover, $\varphi_R(x) \rightarrow 1$ as $R \rightarrow \infty$, and

$$|\underline{f}(\omega^{k+1}) \omega^{k+1} \varphi_R| \leq \nu |\omega^{k+1}|^2 + C_\nu |\omega^{k+1}|^{p+1} \in L^1(\mathbb{R}^N)$$

in view of definition of \underline{f} and Lemma 2.4-(i). Hence, by invoking the dominated convergence theorem we infer that

$$\begin{aligned} \lim_{R \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N} g(\varepsilon x + \varepsilon z_\varepsilon, v_\varepsilon(x + z_\varepsilon)) v_\varepsilon(x + z_\varepsilon) \varphi_R(x) dx &= \lim_{R \rightarrow \infty} \int_{\mathbb{R}^N} \underline{f}(\omega^{k+1}) \omega^{k+1} \varphi_R dx \\ &= \int_{\mathbb{R}^N} \underline{f}(\omega^{k+1}) \omega^{k+1} dx. \end{aligned} \quad (4.27)$$

On the other hand, by using (4.24), Hölder inequality and Lemma 2.1, we can see that

$$\limsup_{R \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \iint_{\mathbb{R}^{2N}} \frac{(v_\varepsilon(x + z_\varepsilon) - v_\varepsilon(y + z_\varepsilon))(\varphi_R(x) - \varphi_R(y))}{|x - y|^{N+2s}} v_\varepsilon(x + z_\varepsilon) dx dy = 0, \quad (4.28)$$

and by applying Fatou's Lemma and (4.24), we get

$$\limsup_{R \rightarrow \infty} \liminf_{\varepsilon \rightarrow 0} \iint_{\mathbb{R}^{2N}} \frac{|v_\varepsilon(x + z_\varepsilon) - v_\varepsilon(y + z_\varepsilon)|^2}{|x - y|^{N+2s}} \varphi_R(y) dx dy \geq \iint_{\mathbb{R}^{2N}} \frac{|\omega^{k+1}(x) - \omega^{k+1}(y)|^2}{|x - y|^{N+2s}} dx dy. \quad (4.29)$$

Taking into account (4.26), (4.27), (4.28) and (4.29), we deduce that

$$\iint_{\mathbb{R}^{2N}} \frac{|\omega^{k+1}(x) - \omega^{k+1}(y)|^2}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} V_0(\omega^{k+1})^2 - \underline{f}(\omega^{k+1}) \omega^{k+1} dx \leq 0. \quad (4.30)$$

By Lemma 2.5 (i)-(ii) and (4.30), we have

$$\iint_{\mathbb{R}^{2N}} \frac{|\omega^{k+1}(x) - \omega^{k+1}(y)|^2}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} (V_0 - \nu)(\omega^{k+1})^2 dx \leq 0.$$

Since $V_0 > \nu$, we infer that $\omega^{k+1} \equiv 0$, which contradicts (4.24).

Then, $\limsup_{\varepsilon \rightarrow 0} |\varepsilon z_\varepsilon| < \infty$ and there exists $x^{k+1} \in \mathbb{R}^N$ such that $\varepsilon z_\varepsilon \rightarrow x^{k+1}$. This and the fact that $\langle J'_\varepsilon(v_\varepsilon), \varphi(\cdot - z_\varepsilon) \rangle \rightarrow 0$ for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, give $\Phi'_{x^{k+1}}(\omega^{k+1}) = 0$. Since $\omega^{k+1} \not\equiv 0$, by Lemma 3.2 (i), follows that $x^{k+1} \in \Omega$.

Step 4: Conclusion.

Let us suppose that $v_0 \neq 0$. Then, we set $y_\varepsilon^1 = 0$, $x^1 = 0$, $\omega^1 = v_0$. If $\|v_\varepsilon - \psi_\varepsilon \omega^1\|_{H_\varepsilon^s} \rightarrow 0$, then (4.5)-(4.8) are satisfied by $0 \in \Omega$, $v_0 \neq 0$ and $\Phi'_0(v_0) = 0$. If $\|v_\varepsilon - \psi_\varepsilon \omega^1\|_{H_\varepsilon^s}$ does not converge to 0, then (4.16) in Step 2 does not occur, and there exists (z_ε) satisfying (4.21) in Step 3. In view of Step 3, there exist x^2, ω^2 verifying (4.22)-(4.25). Then we set $y_\varepsilon^2 = z_\varepsilon$. If $\|v_\varepsilon - \psi_\varepsilon(\omega^1 + \omega^2(\cdot - y_\varepsilon^2))\|_{H_\varepsilon^s} \rightarrow 0$ then (4.5)-(4.8) hold because of $|y_\varepsilon^2 - y_\varepsilon^1| = |z_\varepsilon| \rightarrow \infty$, $\varepsilon y_\varepsilon^2 \rightarrow x^2 \in \Omega$ and $\Phi'_{x^2}(\omega^2) = 0$. Otherwise, we can use Step 2 and 3 to can continue this procedure.

Now, we suppose that $v_0 = 0$. If $\|v_\varepsilon\|_{H_\varepsilon^s} \rightarrow 0$, we have done. Otherwise, the condition (4.16) in

Step 2 does not occur, and we can find (z_ε) satisfying (4.21) in Step 3. By applying Step 3, there exist x^1 and ω^1 verifying (4.22)-(4.25). Thus, we set $y_\varepsilon^1 = z_\varepsilon$.

At this point, we aim to show that this process ends after a finite numbers of steps. Firstly, we show that under the assumptions (4.5)-(4.7) and (4.15)

$$\lim_{\varepsilon \rightarrow 0} \left\| v_\varepsilon - \psi_\varepsilon \sum_{k=1}^n \omega^k(\cdot - y_\varepsilon^k) \right\|_{H_\varepsilon^s}^2 = \lim_{\varepsilon \rightarrow 0} \|v_\varepsilon\|_{H_\varepsilon^s}^2 - \sum_{k=1}^n \|\omega^k\|_{x^k}^2. \quad (4.31)$$

Now

$$\begin{aligned} & \left\| v_\varepsilon - \psi_\varepsilon \sum_{k=1}^n \omega^k(\cdot - y_\varepsilon^k) \right\|_{H_\varepsilon^s}^2 \\ &= \|v_\varepsilon\|_{H_\varepsilon^s}^2 - 2 \sum_{k=1}^n \langle v_\varepsilon, \psi_\varepsilon \omega^k(\cdot - y_\varepsilon^k) \rangle_{H_\varepsilon^s} + \sum_{k,k'} \langle \psi_\varepsilon \omega^k(\cdot - y_\varepsilon^k), \psi_\varepsilon \omega^{k'}(\cdot - y_\varepsilon^{k'}) \rangle_{H_\varepsilon^s}. \end{aligned} \quad (4.32)$$

We show that

$$\langle v_\varepsilon, \psi_\varepsilon \omega^k(\cdot - y_\varepsilon^k) \rangle_{H_\varepsilon^s} \rightarrow \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \omega^k|^2 + V(x^k) (\omega^k)^2 dx = \|\omega^k\|_{x^k}^2. \quad (4.33)$$

In fact

$$\begin{aligned} \langle v_\varepsilon, \psi_\varepsilon \omega^k(\cdot - y_\varepsilon^k) \rangle_{H_\varepsilon^s} &= \iint_{\mathbb{R}^{2N}} \frac{(v_\varepsilon(x + y_\varepsilon^k) - v_\varepsilon(y + y_\varepsilon^k))(\psi_\varepsilon(x + y_\varepsilon^k) - \psi_\varepsilon(y + y_\varepsilon^k))}{|x - y|^{N+2s}} \omega^k(x) dx dy \\ &+ \iint_{\mathbb{R}^{2N}} \frac{(v_\varepsilon(x + y_\varepsilon^k) - v_\varepsilon(y + y_\varepsilon^k))(\omega^k(x) - \omega^k(y))}{|x - y|^{N+2s}} \psi_\varepsilon(y + y_\varepsilon^k) dx dy \\ &+ \int_{\mathbb{R}^N} V(\varepsilon x + \varepsilon y_\varepsilon^k) \psi_\varepsilon(x + y_\varepsilon^k) v_\varepsilon(x + y_\varepsilon^k) \omega^k(x) dx =: (I) + (II) + (III). \end{aligned}$$

We note that arguing as in the proof of (4.13), it is easy to see that $(I) \rightarrow 0$. Concerning (II) , we can observe that

$$\begin{aligned} & \iint_{\mathbb{R}^{2N}} \frac{(v_\varepsilon(x + y_\varepsilon^k) - v_\varepsilon(y + y_\varepsilon^k))(\omega^k(x) - \omega^k(y))}{|x - y|^{N+2s}} \psi_\varepsilon(y + y_\varepsilon^k) dx dy \\ &= \iint_{\mathbb{R}^{2N}} \frac{[(v_\varepsilon(x + y_\varepsilon^k) - v_\varepsilon(y + y_\varepsilon^k)) - (\omega^k(x) - \omega^k(y))]}{|x - y|^{N+2s}} (\omega^k(x) - \omega^k(y)) dx dy \\ &+ \iint_{\mathbb{R}^{2N}} \frac{(\psi_\varepsilon(y + y_\varepsilon^k) - 1)(v_\varepsilon(x + y_\varepsilon^k) - v_\varepsilon(y + y_\varepsilon^k))(\omega^k(x) - \omega^k(y))}{|x - y|^{N+2s}} dx dy =: (II)_1 + (II)_2. \end{aligned}$$

Due to the fact that $v_\varepsilon(\cdot + y_\varepsilon^k) \rightharpoonup \omega^k$ in $H^s(\mathbb{R}^N)$, we obtain that $(II)_1 \rightarrow [\omega^k]^2$. On the other hand, by using Hölder inequality and the fact that $v_\varepsilon(\cdot + y_\varepsilon^k)$ is bounded in $H^s(\mathbb{R}^N)$, we have

$$|(II)_2| \leq C \left(\iint_{\mathbb{R}^{2N}} \frac{(\psi_\varepsilon(y + y_\varepsilon^k) - 1)(\omega^k(x) - \omega^k(y))}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}} \rightarrow 0$$

in view of Lebesgue convergence theorem. Since it is clear that $(III) \rightarrow \int_{\mathbb{R}^N} V(x^k) (\omega^k)^2 dx$, we deduce that (4.33) holds. In similar fashion, we can obtain

$$\langle \psi_\varepsilon \omega^k(\cdot - y_\varepsilon^k), \psi_\varepsilon \omega^{k'}(\cdot - y_\varepsilon^{k'}) \rangle_{H_\varepsilon^s} \rightarrow \begin{cases} 0 & \text{if } k \neq k' \\ \|\omega^k\|_{x^k}^2 & \text{if } k = k' \end{cases} \quad (4.34)$$

Putting together (4.32), (4.33) and (4.34) we can infer that (4.31) holds. Now, (4.31) yields that

$$\sum_{k=1}^n \|\omega^k\|_{x^k}^2 \leq \lim_{\varepsilon \rightarrow 0} \|v_\varepsilon\|_{H_\varepsilon^s}^2$$

and by using Lemma 3.2 (ii) and (4.4) we get

$$\delta_1 n \leq \lim_{\varepsilon \rightarrow 0} \|v_\varepsilon\|_{H_\varepsilon^s}^2 \leq m^2.$$

Therefore, the procedure to find $(y_\varepsilon^k), x^k, \omega^k$ stops after a finite number of steps. Hence, there exists $l \in \mathbb{N} \cup \{0\}$, $(y_\varepsilon^k), x^k, \omega^k$ such that (4.5)-(4.8) are verified. Clearly, (4.9) follows in standard way by (4.5)-(4.8). \square

In the next lemma, we investigate the behavior of c_ε as $\varepsilon \rightarrow 0$.

Lemma 4.2. *Let $(c_\varepsilon)_{\varepsilon \in (0, \varepsilon_1]}$ be the mountain pass value of J_ε defined in (2.18)-(2.19). Then*

$$c_\varepsilon \rightarrow m(0) = \inf_{x \in \mathbb{R}^N} m(x) \text{ per } \varepsilon \rightarrow 0.$$

Proof. By using Lemma 3.1, we can find a path $\gamma \in C([0, 1], H^s(\mathbb{R}^N))$ such that $\gamma(0) = 0$, $\Phi_0(\gamma(1)) < 0$, $\Phi_0(\gamma(t)) \leq m(0)$ for all $t \in [0, 1]$, and

$$\max_{t \in [0, 1]} \Phi_0(\gamma(t)) = m(0).$$

Take $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $\varphi(0) = 1$ and $\varphi \geq 0$, and we set

$$\gamma_R(t)(x) = \varphi\left(\frac{x}{R}\right) \gamma(t)(x).$$

Thus, $\gamma_R(t) \in C([0, 1], H_\varepsilon^s(\mathbb{R}^N))$, $\gamma_R(0) = 0$ and $\Phi_0(\gamma_R(1)) < 0$ for any $R > 1$ sufficiently large. Therefore, $\gamma_R(t) \in \Gamma_\varepsilon$. Fixed $R > 0$, we have $J_\varepsilon(\gamma_R(t)) \rightarrow \Phi_0(\gamma_R(t))$ per $\varepsilon \rightarrow 0$ uniformly in $t \in [0, 1]$. Hence, for any $R > 1$ large enough we get

$$c_\varepsilon \leq \max_{t \in [0, 1]} J_\varepsilon(\gamma_R(t)) \rightarrow \max_{t \in [0, 1]} \Phi_0(\gamma_R(t)) \text{ as } \varepsilon \rightarrow 0.$$

Since

$$\max_{t \in [0, 1]} \Phi_0(\gamma_R(t)) \rightarrow m(0) \text{ as } R \rightarrow \infty,$$

we deduce that $\limsup_{\varepsilon \rightarrow 0} c_\varepsilon \leq m(0)$.

In order to complete the proof, we prove that $\liminf_{\varepsilon \rightarrow 0} c_\varepsilon \geq m(0)$. Let $v_\varepsilon \in H_\varepsilon^s$ be a critical point of $J_\varepsilon(v)$ associated to c_ε . From Lemma 4.1, there exist $\varepsilon_j \rightarrow 0, l \in \mathbb{N} \cup \{0\}$, $(y_{\varepsilon_j}^k), x^k, \omega^k (k = 1, \dots, l)$ satisfying (4.5)-(4.9). If by contradiction $l = 0$, then (4.9) yields $c_{\varepsilon_j} = J_{\varepsilon_j}(v_{\varepsilon_j}) \rightarrow 0$ which contradicts Corollary 2.2. As a consequence, $l \geq 1$ and by using (4.9) we have

$$\liminf_{j \rightarrow \infty} c_{\varepsilon_j} = \sum_{k=1}^l \Phi_{x^k}(\omega^k) \geq \sum_{k=1}^l m(x^k) \geq lm(0) \geq m(0).$$

\square

From Lemma 4.1 and Lemma 4.2, we deduce the following result.

Lemma 4.3. *For any $\varepsilon \in (0, \varepsilon_1]$ we denote by v_ε a critical point of J_ε corresponding to c_ε . Then, for any sequence $\varepsilon_j \rightarrow 0$ we can find a subsequence, still denoted by ε_j , and $y_{\varepsilon_j}, x^1, \omega^1$ such that*

$$\varepsilon_j y_{\varepsilon_j} \rightarrow x^1, \tag{4.35}$$

$$x^1 \in \Lambda' : V(x^1) = \inf_{x \in \Lambda} V(x), \tag{4.36}$$

$$\omega^1(x) \text{ is a least energy solution of } \Phi'_{x^1}(v) = 0, \tag{4.37}$$

$$\|v_{\varepsilon_j} - \psi_{\varepsilon_j} \omega^1(\cdot - y_{\varepsilon_j})\|_{H_{\varepsilon_j}^s} \rightarrow 0, \tag{4.38}$$

$$J_{\varepsilon_j}(v_{\varepsilon_j}) \rightarrow m(x^1) = m(0). \tag{4.39}$$

5. PROOF OF THEOREM 1.1

In this last section we provide the proof of Theorem 1.1. By using Corollary 2.3, we can see that there exists $\varepsilon_1 \in (0, \varepsilon_0]$ such that for any $\varepsilon \in (0, \varepsilon_1]$, there exists a sequence $(v_\varepsilon) \subset H_\varepsilon^s$ of critical points of J_ε corresponding to c_ε . Then, by Lemma 4.3, we know that for any sequence $\varepsilon_j \rightarrow 0$, there exists a subsequence ε_j and $(y_{\varepsilon_j}) \subset \mathbb{R}^N$, $x^1 \in \Omega$, $\omega^1 \in H^s(\mathbb{R}^N) \setminus \{0\}$ satisfying (4.35)-(4.39). By using (4.38) and (2.16), we obtain

$$\|v_{\varepsilon_j} - \psi_{\varepsilon_j} \omega^1(\cdot - y_{\varepsilon_j})\|_{H^s(\mathbb{R}^N)} \rightarrow 0. \quad (5.1)$$

We also note that by (4.38) and (4.31) we get

$$\lim_{j \rightarrow \infty} \|v_{\varepsilon_j}\|_{H_{\varepsilon_j}^s}^2 = \|\omega^1\|_{x^1}^2 \neq 0. \quad (5.2)$$

Let $\tilde{v}_{\varepsilon_j}(x) := v_{\varepsilon_j}(x + y_{\varepsilon_j})$. Arguing as in the proof of (4.13), and recalling the definition of ψ ($\psi = 1$ on Λ) and (4.35), we can see that

$$\begin{aligned} & [\psi_{\varepsilon_j}(\cdot + y_{\varepsilon_j})\omega^1 - \omega^1]^2 \\ & \leq 2 \iint_{\mathbb{R}^{2N}} \frac{|\psi_{\varepsilon_j}(x + y_{\varepsilon_j}) - \psi_{\varepsilon_j}(y + y_{\varepsilon_j})|^2}{|x - y|^{N+2s}} (\omega^1(x))^2 dx dy \\ & + 2 \iint_{\mathbb{R}^{2N}} \frac{|\psi_{\varepsilon_j}(y + y_{\varepsilon_j}) - 1|^2}{|x - y|^{N+2s}} |\omega^1(x) - \omega^1(y)|^2 dx dy \rightarrow 0. \end{aligned}$$

Clearly

$$\int_{\mathbb{R}^N} |\psi_{\varepsilon_j}(x + y_{\varepsilon_j})\omega^1 - \omega^1|^2 dx \rightarrow 0.$$

These two facts, together with (5.1) imply that

$$\|\tilde{v}_{\varepsilon_j} - \omega^1\|_{H^s(\mathbb{R}^N)} \rightarrow 0. \quad (5.3)$$

Now, we prove the following lemma which will be fundamental to study the behavior of the maximum points of the solutions of (1.1).

Lemma 5.1. *There exists $K > 0$ such that*

$$\|\tilde{v}_{\varepsilon_j}\|_{L^\infty(\mathbb{R}^N)} \leq K \text{ for all } j \in \mathbb{N}.$$

Proof. Let $\beta \geq 1$ and $T > 0$, and we introduce the following function

$$\varphi_{T,\beta}(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ t^\beta & \text{if } 0 < t < T \\ \beta T^{\beta-1}(t - T) + T^\beta & \text{if } t \geq T \end{cases}.$$

Since φ is convex and Lipschitz, we can see that for any $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$

$$\begin{aligned} \varphi_{T,\beta}(u) & \in \mathcal{D}^{s,2}(\mathbb{R}^N) \\ (-\Delta)^s \varphi_{T,\beta}(u) & \leq \varphi'_{T,\beta}(u) (-\Delta)^s u. \end{aligned}$$

Now, by using Sobolev inequality, integration by parts, $V > 0$, $\tilde{v}_{\varepsilon_j} \geq 0$, and $|g(x, t)| \leq C(1 + |t|^p)$ with $1 < p < 2_s^* - 1$, we have

$$\begin{aligned}
\|\varphi_{T,\beta}(\tilde{v}_{\varepsilon_j})\|_{L^{2_s^*}(\mathbb{R}^N)}^2 &\leq S_*^{-1} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \varphi_{T,\beta}(\tilde{v}_{\varepsilon_j})|^2 dx \\
&= S_*^{-1} \int_{\mathbb{R}^N} \varphi_{T,\beta}(\tilde{v}_{\varepsilon_j}) (-\Delta)^s \varphi_{T,\beta}(\tilde{v}_{\varepsilon_j}) dx \\
&\leq S_*^{-1} \int_{\mathbb{R}^N} \varphi_{T,\beta}(\tilde{v}_{\varepsilon_j}) \varphi'_{T,\beta}(\tilde{v}_{\varepsilon_j}) (-\Delta)^s \tilde{v}_{\varepsilon_j} dx \\
&= S_*^{-1} \int_{\mathbb{R}^N} \varphi_{T,\beta}(\tilde{v}_{\varepsilon_j}) \varphi'_{T,\beta}(\tilde{v}_{\varepsilon_j}) [-V(\varepsilon_j x + \varepsilon_j y_{\varepsilon_j}) \tilde{v}_{\varepsilon_j} + g(\varepsilon_j x + \varepsilon_j y_{\varepsilon_j}, \tilde{v}_{\varepsilon_j})] dx \\
&\leq C S_*^{-1} \int_{\mathbb{R}^N} \varphi_{T,\beta}(\tilde{v}_{\varepsilon_j}) \varphi'_{T,\beta}(\tilde{v}_{\varepsilon_j}) (1 + \tilde{v}_{\varepsilon_j}^{2_s^*-1}) dx \\
&= C S_*^{-1} \left(\int_{\mathbb{R}^N} \varphi_{T,\beta}(\tilde{v}_{\varepsilon_j}) \varphi'_{T,\beta}(\tilde{v}_{\varepsilon_j}) dx + \int_{\mathbb{R}^N} \varphi_{T,\beta}(\tilde{v}_{\varepsilon_j}) \varphi'_{T,\beta}(\tilde{v}_{\varepsilon_j}) \tilde{v}_{\varepsilon_j}^{2_s^*-1} dx \right)
\end{aligned}$$

where C is a constant independent of β and j . Taking into account $\varphi'_{T,\beta}(\tilde{v}_{\varepsilon_j}) \varphi_{T,\beta}(\tilde{v}_{\varepsilon_j}) \leq \beta \tilde{v}_{\varepsilon_j}^{2\beta-1}$ and $\tilde{v}_{\varepsilon_j} \varphi'_{T,\beta}(\tilde{v}_{\varepsilon_j}) \leq \beta \varphi_{T,\beta}(\tilde{v}_{\varepsilon_j})$, we get

$$\|\varphi_{T,\beta}(\tilde{v}_{\varepsilon_j})\|_{L^{2_s^*}(\mathbb{R}^N)}^2 \leq C\beta \left(\int_{\mathbb{R}^N} \tilde{v}_{\varepsilon_j}^{2\beta-1} dx + \int_{\mathbb{R}^N} (\varphi_{T,\beta}(\tilde{v}_{\varepsilon_j}))^2 \tilde{v}_{\varepsilon_j}^{2_s^*-2} dx \right) \quad (5.4)$$

where C is a constant independent of β and j . We also point out that the last integral in (5.4), is well defined for every $T > 0$ in the definition of $\varphi_{T,\beta}$.

Now, we take $2\beta - 1 = 2_s^*$ in (5.4), and we denote it as

$$\beta_1 = \frac{2_s^* + 1}{2}. \quad (5.5)$$

Let $R > 0$ to be fixed later. By applying Hölder inequality in the last integral in (5.4), we can see that

$$\begin{aligned}
\int_{\mathbb{R}^N} (\varphi_{T,\beta}(\tilde{v}_{\varepsilon_j}))^s \tilde{v}_{\varepsilon_j}^{2_s^*-2} dx &= \int_{\{\tilde{v}_{\varepsilon_j} \leq R\}} (\varphi_{T,\beta}(\tilde{v}_{\varepsilon_j}))^2 \tilde{v}_{\varepsilon_j}^{2_s^*-2} dx + \int_{\{\tilde{v}_{\varepsilon_j} > R\}} (\varphi_{T,\beta}(\tilde{v}_{\varepsilon_j}))^2 \tilde{v}_{\varepsilon_j}^{2_s^*-2} dx \\
&\leq \int_{\{\tilde{v}_{\varepsilon_j} \leq R\}} \frac{(\varphi_{T,\beta}(\tilde{v}_{\varepsilon_j}))^2}{\tilde{v}_{\varepsilon_j}} R^{2_s^*-1} dx + \left(\int_{\mathbb{R}^N} (\varphi_{T,\beta}(\tilde{v}_{\varepsilon_j}))^{2_s^*} dx \right)^{\frac{2}{2_s^*}} \left(\int_{\{\tilde{v}_{\varepsilon_j} > R\}} \tilde{v}_{\varepsilon_j}^{2_s^*} dx \right)^{\frac{2_s^*-2}{2_s^*}}.
\end{aligned} \quad (5.6)$$

Since $\tilde{v}_{\varepsilon_j}$ is bounded in H_ε^s , we can take R sufficiently large such that

$$\left(\int_{\{\tilde{v}_{\varepsilon_j} > R\}} \tilde{v}_{\varepsilon_j}^{2_s^*} dx \right)^{\frac{2_s^*-2}{2_s^*}} \leq \frac{1}{2C\beta_1}.$$

This together with (5.4), (5.5) and (5.6), yield

$$\|\varphi_{T,\beta}(\tilde{v}_{\varepsilon_j})\|_{L^{2_s^*}(\mathbb{R}^N)}^2 \leq 2C\beta_1 \left(\int_{\mathbb{R}^N} \tilde{v}_{\varepsilon_j}^{2_s^*} dx + R^{2_s^*-1} \int_{\mathbb{R}^N} \frac{\varphi_{T,\beta}(\tilde{v}_{\varepsilon_j})^2}{\tilde{v}_{\varepsilon_j}} dx \right). \quad (5.7)$$

By using $\varphi(\tilde{v}_{\varepsilon_j}) \leq \tilde{v}_{\varepsilon_j}^{\beta_1}$ and (5.5), and by taking the limit as $T \rightarrow \infty$ in (5.7), we have

$$\left(\int_{\mathbb{R}^N} \tilde{v}_{\varepsilon_j}^{2_s^*\beta_1} dx \right)^{\frac{2}{2_s^*}} \leq 2C\beta_1 \left(\int_{\mathbb{R}^N} \tilde{v}_{\varepsilon_j}^{2_s^*} dx + R^{2_s^*-1} \int_{\mathbb{R}^N} \frac{\varphi(\tilde{v}_{\varepsilon_j})^2}{\tilde{v}_{\varepsilon_j}} dx \right).$$

which gives

$$\tilde{v}_{\varepsilon_j} \in L^{2_s^* \beta_1}(\mathbb{R}^N). \quad (5.8)$$

Now, we assume that $\beta > \beta_1$. Thus, by using $\varphi(\tilde{v}_{\varepsilon_j}) \leq \tilde{v}_{\varepsilon_j}^\beta$ in the right hand side of (5.4), and by passing to the limit as $T \rightarrow \infty$, we deduce that

$$\begin{aligned} & \left(\int_{\mathbb{R}^N} \tilde{v}_{\varepsilon_j}^{2_s^* \beta} dx \right)^{\frac{2}{2_s^*}} \\ & \leq C\beta \left(\int_{\mathbb{R}^N} \tilde{v}_{\varepsilon_j}^{2\beta-1} dx + \int_{\mathbb{R}^N} \tilde{v}_{\varepsilon_j}^{2\beta+2_s^*-2} dx \right). \end{aligned} \quad (5.9)$$

Set

$$a := \frac{2_s^*(2_s^* - 1)}{2(\beta - 1)} \text{ and } b := 2\beta - 1 - a.$$

By applying Young's inequality with exponents $r = 2_s^*$ and $r' = \frac{2_s^*}{2_s^* - a}$, we can see that

$$\begin{aligned} \int_{\mathbb{R}^N} \tilde{v}_{\varepsilon_j}^{2\beta-1} dx & \leq \frac{a}{2_s^*} \int_{\mathbb{R}^N} \tilde{v}_{\varepsilon_j}^{2_s^*} dx + \frac{2_s^* - a}{2_s^*} \int_{\mathbb{R}^N} \tilde{v}_{\varepsilon_j}^{\frac{2_s^* b}{2_s^* - a}} dx \\ & \leq \int_{\mathbb{R}^N} \tilde{v}_{\varepsilon_j}^{2_s^*} dx + \int_{\mathbb{R}^N} \tilde{v}_{\varepsilon_j}^{2\beta+2_s^*-2} dx \\ & \leq C \left(1 + \int_{\mathbb{R}^N} \tilde{v}_{\varepsilon_j}^{2\beta+2_s^*-2} dx \right). \end{aligned} \quad (5.10)$$

Putting together (5.9) and (5.10), we obtain

$$\left(\int_{\mathbb{R}^N} \tilde{v}_{\varepsilon_j}^{2_s^* \beta} dx \right)^{\frac{2}{2_s^*}} \leq C\beta \left(1 + \int_{\mathbb{R}^N} \tilde{v}_{\varepsilon_j}^{2\beta+2_s^*-2} dx \right). \quad (5.11)$$

As a consequence

$$\left(1 + \int_{\mathbb{R}^N} \tilde{v}_{\varepsilon_j}^{2_s^* \beta} dx \right)^{\frac{1}{2_s^*(\beta-1)}} \leq (C\beta)^{\frac{1}{2(\beta-1)}} \left(1 + \int_{\mathbb{R}^N} \tilde{v}_{\varepsilon_j}^{2\beta+2_s^*-2} dx \right)^{\frac{1}{2(\beta-1)}}. \quad (5.12)$$

Iterating this formula, we obtain

$$\left(1 + \int_{\mathbb{R}^N} \tilde{v}_{\varepsilon_j}^{2_s^* \beta_{k+1}} dx \right)^{\frac{1}{2_s^*(\beta_{k+1}-1)}} \leq (C\beta_{k+1})^{\frac{1}{2(\beta_{k+1}-1)}} \left(1 + \int_{\mathbb{R}^N} \tilde{v}_{\varepsilon_j}^{2_s^* \beta_k} dx \right)^{\frac{1}{2(\beta_k-1)}}. \quad (5.13)$$

where

$$\beta_{k+1} = \left(\frac{2_s^*}{2} \right)^k (\beta_1 - 1) + 1.$$

By setting

$$A_{k+1} := \left(1 + \int_{\mathbb{R}^N} \tilde{v}_{\varepsilon_j}^{2_s^* \beta_{k+1}} dx \right)^{\frac{1}{2_s^*(\beta_{k+1}-1)}}$$

and

$$C_{k+1} := C\beta_{k+1},$$

we can find a constant $c_0 > 0$ independent of k such that

$$A_{k+1} \leq \prod_{m=2}^{k+1} C_k^{\frac{1}{2(\beta_m-1)}} A_1 \leq c_0 A_1.$$

Hence, we can deduce that

$$\|\tilde{v}_{\varepsilon_j}\|_{L^\infty(\mathbb{R}^N)} \leq c_0 A_1 < \infty,$$

uniformly in $j \in \mathbb{N}$, thanks to (5.8) and $\|\tilde{v}_{\varepsilon_j}\|_{H_{\varepsilon_j}^s} \leq M$.

□

By using Lemma 5.1, and the interpolation in L^p spaces, we can see that

$$\tilde{v}_{\varepsilon_j} \rightarrow \omega^1 \text{ in } L^p(\mathbb{R}^N), \text{ for any } p \in (2, \infty), \quad (5.14)$$

$$h_j(x) = g(\varepsilon x + \varepsilon y_{\varepsilon_j}, \tilde{v}_{\varepsilon_j}) \rightarrow f(\omega^1) \text{ in } L^p(\mathbb{R}^N), \text{ for any } p \in (2, \infty). \quad (5.15)$$

Now, we note that $\tilde{v}_{\varepsilon_j}$ satisfies

$$(-\Delta)^s \tilde{v}_{\varepsilon_j} + \tilde{v}_{\varepsilon_j} = \alpha_j \text{ in } \mathbb{R}^N$$

where $\alpha_j(x) = \tilde{v}_{\varepsilon_j}(x) + h_j(x) - V(\varepsilon_j x + \varepsilon_j y_{\varepsilon_j}) \tilde{v}_{\varepsilon_j}(x)$.

In view of (5.14), we can deduce that

$$\alpha_j \rightarrow \omega^1 + f(\omega^1) - V(x^1) \omega^1 \text{ in } L^p(\mathbb{R}^N)$$

for any $p \in [2, \infty)$, so we can find a constant $\kappa > 0$ such that

$$\|\alpha_j\|_{L^\infty(\mathbb{R}^N)} \leq \kappa \text{ for all } j \in \mathbb{N}.$$

Taking into account some results obtained in [27], we know that

$$\tilde{v}_{\varepsilon_j}(x) = (\mathcal{K} * \alpha_j)(x) = \int_{\mathbb{R}^N} \mathcal{K}(x-y) \alpha_j(y) dy,$$

where \mathcal{K} is the Bessel kernel. Then, we can argue as in the proof of Lemma 2.6 in [2], to infer that

$$\tilde{v}_{\varepsilon_j}(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty \quad (5.16)$$

uniformly in $j \in \mathbb{N}$. Now, we are able to prove that $\tilde{v}_{\varepsilon_j}$ is a solution to (1.1) for sufficiently small $\varepsilon_j > 0$. By using the fact that $\varepsilon_j y_{\varepsilon_j} \rightarrow x^1 \in \Lambda'$, there exists $r > 0$ such that for some subsequence, still denoted by itself, we have

$$B_r(\varepsilon_j y_{\varepsilon_j}) \subset \Lambda' \text{ for all } j \in \mathbb{N}.$$

By setting $\Lambda'_\varepsilon = \frac{\Lambda'}{\varepsilon}$, we can see that

$$B_{\frac{r}{\varepsilon_j}}(y_{\varepsilon_j}) \subset \Lambda'_{\varepsilon_j} \text{ for all } j \in \mathbb{N}$$

which yields

$$\mathbb{R}^N \setminus \Lambda'_{\varepsilon_j} \subset \mathbb{R}^N \setminus B_{\frac{r}{\varepsilon_j}}(y_{\varepsilon_j}) \text{ for all } j \in \mathbb{N}.$$

By using (5.16), there exists $R > 0$ such that

$$\tilde{v}_{\varepsilon_j}(x) < r_\nu \text{ for all } |x| \geq R, j \in \mathbb{N}$$

so that

$$v_{\varepsilon_j}(x) = \tilde{v}_{\varepsilon_j}(x - y_{\varepsilon_j}) < r_\nu \text{ for all } x \in \mathbb{R}^N \setminus B_{\frac{R}{\varepsilon_j}}(y_{\varepsilon_j}), j \in \mathbb{N}.$$

On the other hand, there exists $j_0 \in \mathbb{N}$ such that

$$\mathbb{R}^N \setminus \Lambda'_{\varepsilon_j} \subset \mathbb{R}^N \setminus B_{\frac{r}{\varepsilon_j}}(y_{\varepsilon_j}) \subset \mathbb{R}^N \setminus B_R(y_{\varepsilon_j}) \text{ for all } j \geq j_0.$$

Hence

$$v_{\varepsilon_j}(x) < r_\nu \text{ for all } x \in \mathbb{R}^N \setminus \Lambda'_{\varepsilon_j}, j \geq j_0. \quad (5.17)$$

Now, up to a subsequence, we may assume that

$$\|v_{\varepsilon_j}\|_{L^\infty(B_R(y_{\varepsilon_j}))} \geq r_\nu \text{ for all } j \geq j_0. \quad (5.18)$$

Otherwise, if this is not the case, we have $\|v_{\varepsilon_j}\|_{L^\infty(\mathbb{R}^N)} < r_\nu$, and taking into account the definition of g and our choice of r_ν , we get

$$g(\varepsilon_j x, v_{\varepsilon_j}) v_{\varepsilon_j} = f(v_{\varepsilon_j}) v_{\varepsilon_j} \leq \nu v_{\varepsilon_j}^2.$$

Then, by $\langle J'_{\varepsilon_j}(v_{\varepsilon_j}), v_{\varepsilon_j} \rangle = 0$, we can deduce that

$$\|v_{\varepsilon_j}\|_{H_{\varepsilon_j}^s}^2 = \int_{\mathbb{R}^N} f(v_{\varepsilon_j}) v_{\varepsilon_j} dx \leq \nu \int_{\mathbb{R}^N} v_{\varepsilon_j}^2 dx$$

which implies that $\lim_{j \rightarrow \infty} \|v_{\varepsilon_j}\|_{H_{\varepsilon_j}^s}^2 = 0$, which is a contradiction in view of (5.2). Therefore, putting together (5.19) and (5.20), we deduce that the maximum points $z_{\varepsilon_j} \in \mathbb{R}^N$ of v_{ε_j} belong to $B_R(y_{\varepsilon_j})$. Hence $z_{\varepsilon_j} = y_{\varepsilon_j} + \bar{z}_{\varepsilon_j}$, for some $\bar{z}_{\varepsilon_j} \in B_R(0)$. Since the solution of our problem (1.1) is given by $u_{\varepsilon_j}(x) = v_{\varepsilon_j}(\frac{x}{\varepsilon_j})$, we can conclude that the maximum point x_{ε_j} of u_{ε_j} is $x_{\varepsilon_j} := \varepsilon_j y_{\varepsilon_j} + \varepsilon_j \bar{z}_{\varepsilon_j}$. Being $(\bar{z}_{\varepsilon_j}) \subset B_R(0)$ is bounded and $\varepsilon_j y_{\varepsilon_j} \rightarrow x^1 \in \Lambda'$ we obtain

$$\lim_{j \rightarrow \infty} V(x_{\varepsilon_j}) = V(x^1) = \inf_{x \in \Lambda} V(x).$$

Therefore, we have proved that there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, (1.1) admits a positive solution $u_\varepsilon(x) = v_\varepsilon(\frac{x}{\varepsilon})$ satisfying (1) of Theorem 1.1. Finally, we prove that (2) holds. By using Lemma 4.3 in [27], we know that there exists w such that

$$0 < w(x) \leq \frac{C}{1 + |x|^{N+2s}}, \quad (5.19)$$

and

$$(-\Delta)^{\frac{s}{2}} w + \frac{V_0}{2} w \geq 0 \text{ in } \mathbb{R}^N \setminus B_{R_1} \quad (5.20)$$

for some suitable $R_1 > 0$. In view of (5.16), we know that $\tilde{v}_{\varepsilon_j} \rightarrow 0$ as $|x| \rightarrow \infty$ uniformly in j . This, (f2) and the definition of g , imply that for some $R_2 > 0$ sufficiently large, we get

$$\begin{aligned} (-\Delta)^{\frac{s}{2}} \tilde{v}_{\varepsilon_j} + \frac{V_0}{2} \tilde{v}_{\varepsilon_j} &= g(\varepsilon_j x + \varepsilon_j y_{\varepsilon_j}, \tilde{v}_{\varepsilon_j}) - \left(V - \frac{V_0}{2} \right) \tilde{v}_{\varepsilon_j} \\ &\leq g(\varepsilon_j x + \varepsilon_j y_{\varepsilon_j}, \tilde{v}_{\varepsilon_j}) - \frac{V_0}{2} \tilde{v}_{\varepsilon_j} \leq 0 \text{ in } \mathbb{R}^N \setminus B_{R_2}. \end{aligned} \quad (5.21)$$

Choose $R_3 = \max\{R_1, R_2\}$, and we set

$$a = \inf_{B_{R_3}} w > 0 \text{ and } \tilde{w}_{\varepsilon_j} = (b+1)w - a\tilde{v}_{\varepsilon_j}. \quad (5.22)$$

where $b = \sup_{j \in \mathbb{N}} \|\tilde{v}_{\varepsilon_j}\|_{L^\infty(\mathbb{R}^N)} < \infty$. Now, we prove that

$$\tilde{w}_{\varepsilon_j} \geq 0 \text{ in } \mathbb{R}^N. \quad (5.23)$$

We first note that

$$\tilde{w}_{\varepsilon_j} \geq ba + w - ba > 0 \text{ in } B_{R_3}, \quad (5.24)$$

$$(-\Delta)^{\frac{s}{2}} \tilde{w}_{\varepsilon_j} + \frac{V_0}{2} \tilde{w}_{\varepsilon_j} \geq 0 \text{ in } \mathbb{R}^N \setminus B_{R_3}. \quad (5.25)$$

We argue by contradiction, and we assume that there exists a sequence $(\bar{x}_{j,n}) \subset \mathbb{R}^N$ such that

$$\inf_{x \in \mathbb{R}^N} \tilde{w}_{\varepsilon_j}(x) = \lim_{n \rightarrow \infty} \tilde{w}_{\varepsilon_j}(\bar{x}_{j,n}) < 0. \quad (5.26)$$

By using (5.16) and the definition of $\tilde{w}_{\varepsilon_j}$, it is clear that $\tilde{w}_{\varepsilon_j}(x) \rightarrow 0$ as $|x| \rightarrow \infty$, uniformly in $j \in \mathbb{N}$. Thus, we can deduce that $(\bar{x}_{j,n})$ is bounded, and, up to subsequence, we may assume that there exists $\bar{x}_j \in \mathbb{R}^N$ such that $\bar{x}_{j,n} \rightarrow \bar{x}_j$ as $n \rightarrow \infty$. Thus, from (5.26), we get

$$\inf_{x \in \mathbb{R}^N} \tilde{w}_{\varepsilon_j}(x) = \tilde{w}_{\varepsilon_j}(\bar{x}_j) < 0. \quad (5.27)$$

By using the minimality of \bar{x}_j and the representation formula for the fractional Laplacian, we can see that

$$(-\Delta)^s \tilde{w}_{\varepsilon_j}(\bar{x}_j) = C(N, s) \int_{\mathbb{R}^N} \frac{2\tilde{w}_{\varepsilon_j}(\bar{x}_j) - \tilde{w}_{\varepsilon_j}(\bar{x}_j + \xi) - \tilde{w}_{\varepsilon_j}(\bar{x}_j - \xi)}{|\xi|^{N+2s}} d\xi \leq 0. \quad (5.28)$$

Taking into account (5.24) and (5.26), we can infer that $\bar{x}_j \in \mathbb{R}^N \setminus B_{R_3}$. This together with (5.27) and (5.28), yield

$$(-\Delta)^{\frac{s}{2}} \tilde{w}_{\varepsilon_j}(\bar{x}_j) + \frac{V_0}{2} \tilde{w}_{\varepsilon_j}(\bar{x}_j) < 0,$$

which contradicts (5.25). Thus (5.23) holds, and by using (5.19) we get

$$\tilde{v}_{\varepsilon_j}(x) \leq \frac{\tilde{C}}{1 + |x|^{N+2s}} \text{ for all } j \in \mathbb{N}, x \in \mathbb{R}^N, \quad (5.29)$$

for some $\tilde{C} > 0$. Since $u_{\varepsilon_j}(x) = v_{\varepsilon_j}(\frac{x}{\varepsilon_j}) = \tilde{v}_{\varepsilon_j}(\frac{x}{\varepsilon_j} - y_{\varepsilon_j})$ and $x_{\varepsilon_j} = \varepsilon_j y_{\varepsilon_j} + \varepsilon_j \bar{z}_{\varepsilon_j}$, from (5.29) we obtain for any $x \in \mathbb{R}^N$

$$\begin{aligned} u_{\varepsilon_j}(x) &= v_{\varepsilon_j}\left(\frac{x}{\varepsilon_j}\right) = \tilde{v}_{\varepsilon_j}\left(\frac{x}{\varepsilon_j} - y_{\varepsilon_j}\right) \\ &\leq \frac{\tilde{C}}{1 + \left|\frac{x}{\varepsilon_j} - y_{\varepsilon_j}\right|^{N+2s}} \\ &= \frac{\tilde{C}\varepsilon_j^{N+2s}}{\varepsilon_j^{N+2s} + |x - \varepsilon_j y_{\varepsilon_j}|^{N+2s}} \\ &\leq \frac{\tilde{C}\varepsilon_j^{N+2s}}{\varepsilon_j^{N+2s} + |x - x_{\varepsilon_j}|^{N+2s}}. \end{aligned}$$

This ends the proof of Theorem 1.1.

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